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COUPLING OF DARCY-FORCHHEIMER AND COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH HEAT TRANSFER*

M. AMARA †, D. CAPATINA‡, AND L. LIZAIK §

Abstract. This paper is devoted to the coupling of a 2D reservoir model with a 1.5D vertical wellbore model, both written in axisymmetric form. The physical problems are respectively described by the Darcy-Forchheimer and the compressible Navier-Stokes equations, together with an exhaustive energy equation. Each model was previously studied and its finite element discretization was validated. The two weak problems are bound together by means of transmission conditions at the perforations, yielding a non standard mixed formulation. A technical analysis is then carried out and the well-posedness of the time-discretized coupled problem, in both the continuous and the discrete cases, is established. Numerical tests including physical cases are presented, validating the coupled code.

Key words. Petroleum wellbore and reservoir, Darcy-Forchheimer, Navier-Stokes, mixed finite elements, multiscale coupling

AMS subject classifications. 65M60, 80A20, 76S05

Introduction. Thermometric studies in petroleum wellbores and reservoirs have been largely developed in the past years, since they allow to better characterize reservoirs. By installing captors such as optical fiber sensors in the well, it is now possible to measure the temperature continuously in time and all along the well. Using these recordings as well as a flowrate history at the bottom of the well, the hope is twofold: to predict the flow repartition between each producing layer and to estimate the virgin reservoir temperature.

In order to solve these inverse problems, one first needs to develop a forward model describing the flow of a compressible fluid in a petroleum reservoir (porous medium) and a well (fluid medium), from both a dynamic and a thermal point of view. We only consider here a single phase flow.

There exist many simulators dedicated to reservoir and wellbore modeling but most of them are either isothermal or neglect certain physical phenomena, which play an important role when small variations of temperature are to be interpreted.

A reservoir model, written in cylindrical coordinates and consisting of the Darcy-Forchheimer equation coupled with an exhaustive energy balance, has already been studied in [1]. The energy equation notably includes the temperature effects due to the decompression of the fluid (Joule-Thomson effect) and the frictional heating that occurs in the formation. The problem was time-discretized by means of Euler's implicit scheme, leading to a linearized system at each time step.

A vertical wellbore model, also written in axisymmetric form and based on the compressible Navier-Stokes equations coupled with an energy equation, was introduced and analyzed in [2]. In order to take into account the privileged direction of the flow and to reduce the computational cost, a 1.5D model was derived as a conforming approximation of the 2D axisymmetric one by constructing an explicit solution in terms of the radial coordinate r. The nonlinear time-discretized problem was then solved by means of a fixed point method with respect to the density.

*This work is supported by Total

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Both proposed models were separately validated from a numerical and a physical point of view. This paper is devoted to their coupling.

On the one hand, we have established existence and uniqueness of the solution of the time-discretized coupled problem, in both the continuous and the discrete cases. The time-discretized coupled problem is solved globally and takes into account the convective terms as well as the transmission conditions between the reservoir and the wellbore. On the other hand, we have carried out several numerical tests (including realistic applications) for the coupled problem which validate our code.

Let us note that our coupled problem is different from those currently considered in the literature (Cf. for instance [9]). First of all, the two models don't have neither the same dimensions nor the same number of unknown functions. Morever, the density is not constant in the two media. Finally, the energetic aspect is taken here into account, which is not the case in most papers devoted to the coupling of Stokes (or Navier-Stokes) and Darcy equations.

In order to achieve the coupling, adequate transmission conditions at the perforations are imposed and next dualized by means of Lagrange multipliers. We finally obtain at each time step a mixed weak formulation whose operator is mathematically non standard, since it can be written as :

$$\begin{bmatrix} \mathcal{A} & \mathcal{I} \\ \mathcal{J} & 0 \end{bmatrix}, \text{ with } \mathcal{A} = \begin{bmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{B}^T & -\mathtt{C} \end{bmatrix}.$$

Here above, $\mathtt{A}$ and $\mathtt{C}$ are non-symmetric while the unknowns and the test-functions belong to different spaces.

The operator $\mathcal{A}$ was shown to satisfy an inf-sup condition, yielding the uniqueness of the solution thanks to Babuška's theorem. However, at this stage, we couldn't prove the second inf-sup condition which ensures the existence.

In order to take into account recorded flowrates at the pipe's surface, a global solving of the coupled problem is envisaged. Concerning the spatial discretization, we approximate the heat and mass fluxes by the lowest-order Raviart-Thomas elements, the pressure and the temperature by P_0 elements, the fluid's velocity by Q_1 continuous elements while the Lagrange multipliers at the interface are taken piecewise constant. The convective terms are treated by appropriated upwind schemes. The well-posedness of the discrete problem was established and finally, the existence of a solution for the continuous problem was also proved by means of a Galerkin method.

Numerical tests including real cases are presented, in order to validate the developed code. The behavior of the solution with respect to mesh refinement is also studied and comparisons with the results obtained separately by the two models are carried out.

The outline of the paper is as follows. In Section 1, we briefly recall the reservoir model while Section 2 focuses on the wellbore model. Sections 3 and 4 contain the main results of the paper, since they are devoted to the analysis of the continuous, respectively discrete coupled problems. Numerical tests are presented in Section 5.

As future works, besides developing an approach to solve the cited inverse problems, we intend to extend this work to multiphase flows. To do so, a black-oil model is retained for the reservoir but one has to tackle a modeling difficulty related to our non standard energy equation. Furthermore, one can also envisage to treat the more general case of deviated wellbores.

Besides, there are several open questions in the mathematical analysis which can be further addressed. For the sake of simplicity, one could consider a model problem as simple as possible so as to be interesting (such as the coupling between Darcy and Stokes equations with varying densities and varying permeabilities, to which an energy balance could next be added). Then some issues (to cite only a few) are : the analysis of the time-discretization and its convergence, the study of the nonlinear problem at each time-step, the derivation of error estimates, the treatment of non-matching grids at the interface etc.

Let us end this section by introducing some notation. We agree to write the vectors in bold letters and the tensors in underlined bold letters. As usually, for a given domain ω of $\mathbb{R}^n$ we shall denote by $L^2(\omega)$ the space of square integrable functions for the Lebesgue measure on ω and we put:

$$\begin{array}{l} H^1(\omega) = \left\{u \in L^2(\omega)\,;\; \nabla u \in \left(L^2(\omega)\right)^n\right\}, \\ H(div,\omega) = \left\{\mathbf{u} \in \left(L^2(\omega)\right)^n;\; div\mathbf{u} \in L^2(\omega)\right\}, \\ \mathbf{H}(div,\omega) = \left(H(div,\omega)\right)^n, \quad \mathbf{L}^2(\omega) = \left(L^2(\omega)\right)^n. \end{array}$$

For the sake of clarity, we shall denote by Ω_1 the 2D domain occupied by the porous medium, by Ω_2 the 2D domain of the fluid. For a given boundary $\Gamma \subset \partial\omega$, we denote by $\langle \cdot,\cdot \rangle_\Gamma$ the duality product between $H^{1/2}_{00}(\Gamma)$ and its dual space $H^{-1/2}(\Gamma)$; we recall that $H^{1/2}_{00}(\Gamma)$ is the space of traces on Γ of functions in $H^1(\omega)$ which vanish on $\partial\omega \setminus \Gamma$. The letter c denotes any positive constant independent of both the time and the space discretizations. For any affine set V^*, we agree to denote by V^0 the associated vectorial space.

1. 2D Reservoir model. The studied domain (see for instance Figure 1.1) is a cylindrical petroleum well, delimited by a casing and surrounded by a cement layer and a reservoir, assumed to be a porous medium with an axisymmetric geometry. The two domains communicate through the perforations Σ. For instance, the reservoir can be multi-layered, each layer being characterized by its own physical properties and being saturated with both a mobile single phase fluid and a residual formation water.

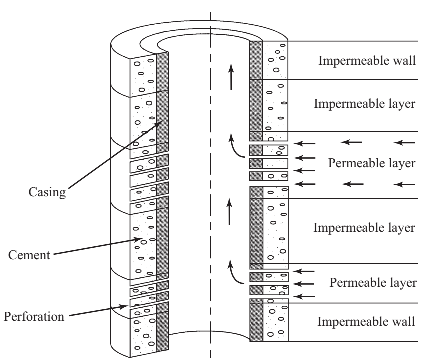


Fig. 1.1. *Geometry of a wellbore surrounded by a reservoir.*

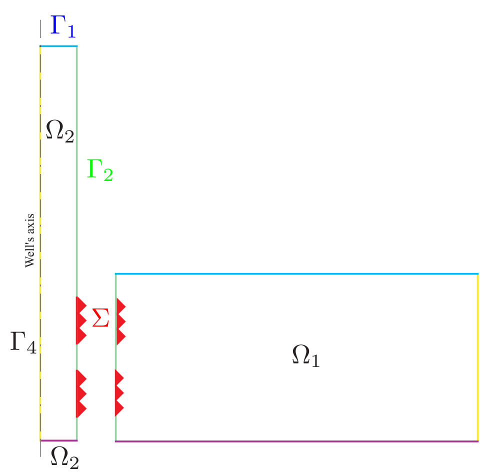


Fig. 1.2. *Boundaries of the domain*

The mass conservation can be written as follows :

$$\phi\frac{\partial\rho}{\partial t} + div\mathbf{G} = 0,$$

where ρ is the fluid's density, $\mathbf{G} = \rho\mathbf{v}$ denotes the specific flux with $\mathbf{v}$ the Darcy velocity and ϕ is the porosity.

Due to the high filtration velocity which can arise around gas wells, a quadratic term in the standard Darcy equation is introduced (Cf. [18]), in order to take into account the kinematic energy losses. We thus get :

$$\rho^{-1}(\mu\underline{\mathbf{K}}^{-1}\mathbf{G} + F|\mathbf{G}|\,\mathbf{G}) + \nabla p = -\rho\mathbf{g},$$

where F represents the Forchheimer coefficient, $\underline{\mathbf{K}} = \begin{bmatrix} k_h & 0 \\ 0 & k_v \end{bmatrix}$ the permeability tensor (with k_h, k_v the horizontal, respectively vertical permeabilities), μ the viscosity of the fluid, p the pressure and $\mathbf{g}$ the gravitational acceleration.

We next consider an energy equation (Cf. [11]) which takes into account, besides the convection and the diffusion, viscous dissipation and compressibility effects :

$$(\rho c)_* \frac{\partial T}{\partial t} + \rho^{-1}(\rho c)_f \mathbf{G}\cdot\nabla T - div\mathbf{q} - \phi\beta T\frac{\partial p}{\partial t} - \rho^{-1}(\beta T - 1)\mathbf{G}\cdot\nabla p = 0,$$

where β is the expansion coefficient, $(\rho c)_*$ characterizes the heat capacity of a virtual medium, equivalent to the fluid and the porous matrix, while $(\rho c)_f$ symbolizes only the fluid properties. The heat flux is represented by $\mathbf{q} = \lambda\nabla T$ where λ is the thermal conductivity and T the temperature.

Finally, we close the system by considering the Peng-Robinson state equation (Cf. [14]), which is simply written here as follows : $\rho = \rho(p, T)$.

One still has to add initial conditions for p and T, as well as boundary conditions. An impermeability condition $\mathbf{G}\cdot\mathbf{n} = 0$ is imposed on the top, the bottom and the non perforated internal boundary while the pressure is prescribed on the external boundary. The geothermal gradient is imposed on the bottom and on the top, an adiabatic condition $\mathbf{q}\cdot\mathbf{n} = 0$ is set on the non perforated internal boundary and the temperature is given on the external boundary. On the perforations Σ, one can impose $\mathbf{G}\cdot\mathbf{n}$ or its dual variable p , respectively $\mathbf{q}\cdot\mathbf{n}$ or T.

In what follows, for the sake of clarity we shall denote by Υ_p, Υ_T, Υ_G and Υ_q the boundaries where a pressure p^*, a temperature T^*, a normal specific flux G^*, respectively a normal heat flux q^* are given.

Due to the particular geometry, the previous nonlinear system was next written in 2D axisymmetric form on the rectangular domain defined by :

$$\Omega_1 = \{(r, z)\ ;\ R \le r \le R_\infty\ ,\ z \in [z_{min},\ z_{max}]\}$$

where R is the radius of the well and R_∞ the reservoir's one. The time-discretization is achieved by means of Euler's implicit scheme; by linearizing the convective terms, we obtain at each time step the following linear system :

$$\begin{cases} \frac{1}{r}\underline{\mathbf{M}}\mathbf{G} + \nabla p = -\rho^{n-1}\mathbf{g} \\ \frac{1}{r\lambda}\mathbf{q} - \nabla T = 0 \\ r\frac{a}{\Delta t}p - r\frac{b}{\Delta t}T + div\mathbf{G} = r\frac{a}{\Delta t}p^{n-1} - r\frac{b}{\Delta t}T^{n-1} \\ r\frac{d}{\Delta t}T + \kappa\mathbf{G}^{n-1}\cdot\nabla T - r\frac{f}{\Delta t}p + l\mathbf{G}^{n-1}\cdot\nabla p - div\mathbf{q} = r\frac{d}{\Delta t}T^{n-1} - r\frac{f}{\Delta t}p^{n-1} \end{cases} \tag{1.1}$$

where now $\nabla = \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right)^t$ and $div\mathbf{v} = \nabla \cdot \mathbf{v}$. The thermodynamic coefficients $a,\ b,\ d,\ k,\ l,\ f$ are computed at t^{n-1}, the tensor $\underline{\mathbf{M}}$ defined by

$$\underline{\mathbf{M}} = \frac{1}{\rho^{n-1}}(\mu\underline{\mathbf{K}}^{-1} + \frac{F}{r}\left|\mathbf{G}^{n-1}\right|\underline{\mathbf{I}})$$

is bounded and positive definite, the thermal conductivity satisfies $\lambda_1 \geq \lambda \geq \lambda_0 > 0$.

In order to study problem (1.1), we write a mixed variational formulation. For this purpose, let us denote by $\mathbf{V} = (\mathbf{G}, \mathbf{q})$ the vector unknowns, by $\mathbf{s} = (p, T)$ the scalar ones and let us introduce :

$$\mathbf{H}^*(div, \Omega_1) = \{\mathbf{V} = (\mathbf{G}, \mathbf{q}) \in \mathbf{H}(div, \Omega_1);\ \mathbf{G}\cdot\mathbf{n} = G^* \ \text{ on } \ \Upsilon_G,\ \mathbf{q}\cdot\mathbf{n} = q^* \ \text{ on } \ \Upsilon_q\}.$$

Then, the time-discretized problem has the following weak form :

$$\begin{cases} \text{Find } (\mathbf{V}, \mathbf{s}) \in \mathbf{H}^*(div, \Omega_1) \times \mathbf{L}^2(\Omega_1) \\ A(\mathbf{V}, \mathbf{V}') + B(\mathbf{s}, \mathbf{V}') = F_1(\mathbf{V}'), & \forall\ \mathbf{V}' \in \mathbf{H}^0(div, \Omega_1), \\ B(\mathbf{s}', \mathbf{V}) - C(\mathbf{s}, \mathbf{s}') - \alpha D(\mathbf{s}, \mathbf{s}') = F_2(\mathbf{s}'), & \forall\ \mathbf{s}' \in \mathbf{L}^2(\Omega_1), \end{cases} \tag{1.2}$$

where:

$$A(\mathbf{V}, \mathbf{V}') = \int_{\Omega_1} \frac{1}{r}\underline{\mathbf{M}}\mathbf{G}\cdot\mathbf{G}'dx + \int_{\Omega_1} \frac{1}{r\lambda}\mathbf{q}\mathbf{q}'dx,$$

$$B(\mathbf{s}, \mathbf{V}') = -\int_{\Omega_1} p\,div\mathbf{G}'dx + \int_{\Omega_1} T\,div\mathbf{q}'dx,$$

$$C(\mathbf{s}, \mathbf{s}') = \int_{\Omega_1} r\frac{a}{\Delta t}pp'dx - \int_{\Omega_1} r\frac{b}{\Delta t}Tp'dx + \int_{\Omega_1} r\frac{d}{\Delta t}TT'dx - \int_{\Omega_1} r\frac{f}{\Delta t}pT'dx,$$

$$D(\mathbf{s}, \mathbf{s}') = \int_{\Omega_1} \kappa\mathbf{G}^{n-1}\cdot\nabla TT'dx + \int_{\Omega_1} l\mathbf{G}^{n-1}\cdot\nabla pT'dx,$$

$$F_1(\mathbf{V}') = -\int_{\Omega_1} \rho^{n-1}\mathbf{g}\cdot\mathbf{G}'dx - \langle\mathbf{G}'\cdot\mathbf{n}, p^*\rangle_{\partial\Omega_1} + \langle\mathbf{q}'\cdot\mathbf{n}, T^*\rangle_{\partial\Omega_1},$$

$$F_2(\mathbf{s}') = \int_{\Omega_1} \frac{r}{\Delta t}\left(ap^{n-1} - bT^{n-1}\right)p'dx + \int_{\Omega_1} \frac{r}{\Delta t}\left(dT^{n-1} - fp^{n-1}\right)T'dx$$

and where the parameter α equals 1 for the complete problem, respectively 0 for the problem without convection. Problem (1.2) can be equivalently written as follows :

$$\begin{cases} \text{Find } \ x_1 \in X_1^* \\ \mathcal{A}_1(x_1, x_1') = \mathcal{F}_1(x_1'),\ \forall x_1' \in X_1^0 \end{cases} \tag{1.3}$$

where $x_1 = (\mathbf{V}, \mathbf{s})$ and :

$$\mathcal{A}_1 = \begin{bmatrix} A & B \\ B^T & -C - \alpha D \end{bmatrix}, \qquad \mathcal{F}_1 = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

Here above, we have employed the notation :

$$X_1 = \mathbf{H}(div, \Omega_1) \times \mathbf{L}^2(\Omega_1), \quad X_1^* = \mathbf{H}^*(div, \Omega_1) \times \mathbf{L}^2(\Omega_1).$$

In the case $\alpha = 0$, the problem was shown to have a unique solution, under some boundedness and positivity conditions on the thermodynamic coefficients. The proof is based on an extension of the Babuška-Brezzi theorem (Cf. [17]) to the case A positive, symmetric and elliptic on $Ker\ B$, C positive but non-symmetric and B satisfying an inf-sup condition. Finally, the well-posedness of the complete problem with convection (i.e. $\alpha = 1$) was established by means of Fredholm's alternative, for Δt sufficiently small. We refer to [1] for the detailed proofs.

2. 1.5D Wellbore Model. The governing kinematic equations in the fluid medium are the mass conservation law and the Navier-Stokes equations with a source term which takes into account the friction at the pipe's surface. We also consider the energy equation and we close the system by the same Peng-Robinson state equation.

As for the reservoir, the problem is written in 2D axisymmetric form, depending only on the cylindrical coordinates (r, z). Thus, the 2D domain merely consists of :

$$\Omega_2 = \{(r, z)\ ;\ 0 \le r \le R,\ z \in I\}$$

where $I = [z_1, z_2]$. In practice, $R \simeq 4inch$ while the length of the pipe can attend several thousands meters. Our problem is then described by :

$$\begin{cases} \frac{\partial}{\partial t}(r\rho) + \nabla \cdot (r\rho\mathbf{u}) = 0 \\ \frac{\partial}{\partial t}(r\rho u_r) + \nabla \cdot (ru_r\rho\mathbf{u}) + r\frac{\partial p}{\partial r} - \frac{\partial}{\partial r}(r\tau_{rr}) - \frac{\partial}{\partial z}(r\tau_{zr}) + \tau_{\theta\theta} + r\kappa\rho|\mathbf{u}|u_r = 0 \\ \frac{\partial}{\partial t}(r\rho u_z) + \nabla \cdot (ru_z\rho\mathbf{u}) + r\frac{\partial p}{\partial z} - \frac{\partial}{\partial r}(r\tau_{rz}) - \frac{\partial}{\partial z}(r\tau_{zz}) + r\rho g + r\kappa\rho|\mathbf{u}|u_z = 0 \\ \frac{\partial}{\partial t}(r\rho E) + \nabla \cdot (r(\rho E + p)\mathbf{u}) - \nabla \cdot (r\underline{\tau}\mathbf{u}) - \nabla \cdot (r\lambda\nabla T) + r\rho g u_z = 0 \\ \rho = \rho(p, T) \end{cases} \tag{2.1}$$

where $\mathbf{u} = (u_r, u_z)$ and the tensor $\underline{\tau}$ is defined (cf. for instance [12]) by :

$$\tau_{rr} = 2\mu\frac{\partial u_r}{\partial r} - \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right), \quad \tau_{rz} = \tau_{zr} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right),$$

$$\tau_{zz} = 2\mu\frac{\partial u_z}{\partial z} - \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right), \quad \tau_{\theta\theta} = 2\mu\frac{u_r}{r} - \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right).$$

Here above, $E = c_vT + \frac{|\mathbf{u}|^2}{2}$ is the total energy, c_v is the specific heat and κ is a positive coefficient depending on the diameter of the pipe. We assume in what follows that $\rho_1 \ge \rho(z) \ge \rho_0 > 0$ a.e. on Σ and $\lambda_1 \ge \lambda \ge \lambda_0 \ge 0$ a.e. in Ω_2.

A 2D computation confirmed that the flow in the wellbore is essentially vertical (cf. [6]). In order to take into account the flow privileged direction, the particular geometry of the domain, as well as the supply at the perforations, a 1.5D modeling was proposed in [2]. Thus, calculations are lightened and moreover, one avoids any numerical instability due to the large aspect ratio of any 2D grid.

Let us next recall the derivation of the simplified wellbore model. One first introduces two conservative variables (the specific flux $\mathbf{G} = \rho\mathbf{u}$ and the heat flux $\mathbf{q} = \lambda\nabla T$) and a time discretization which yields, at each time step, a nonlinear system. A fixed point method with respect to the density is then applied and the proposed algorithm consists in solving, for a given ρ, three decoupled problems :

$$div(r\mathbf{G}) = -r\frac{\rho - \rho^{n-1}}{\Delta t}, \tag{2.2}$$

$$\begin{cases} div(r\mathbf{u}) = \dfrac{1}{\rho}(div(r\mathbf{G}) - \dfrac{r}{\rho}\mathbf{G} \cdot \nabla\rho) \\ r\rho\frac{\mathbf{u}}{\Delta t} + r\mathbf{G} \cdot \nabla\mathbf{u} + r\nabla p - div(r\underline{\tau}) + \tau_{\theta\theta}\mathbf{e}_r + r\kappa|\mathbf{G}|\mathbf{u} = r\rho\mathbf{g} + r\rho\frac{\mathbf{u}^{n-1}}{\Delta t} \end{cases} \tag{2.3}$$

$$\left\{\begin{array}{c} rc_v\left(\rho\frac{T}{\Delta t}+\mathbf{G}\cdot\nabla T\right)-div(r\mathbf{q}) \\ \\ = r\rho c_v\frac{T^{n-1}}{\Delta t}-\frac{1}{2}r\left(\rho\frac{|\mathbf{u}|^2-|\mathbf{u}^{n-1}|^2}{\Delta t}+\mathbf{G}\cdot\nabla(|\mathbf{u}|^2)\right)-div(rp\mathbf{u})+div(r\underline{\tau}\mathbf{u})+r\mathbf{g}\cdot\mathbf{G} \\ \\ \mathbf{q}=\lambda\nabla T. \end{array}\right. \tag{2.4}$$

Finally, the density is updated by means of a thermodynamic module (available at TOTAL) and one loops until convergence is achieved.

Remark 1. *The first equation of (2.3) translates the fact that $div(r\mathbf{u}) = div(\frac{r}{\rho}\mathbf{G})$ while in the other equations we have simply substituted $\rho\mathbf{u}$ by $\mathbf{G}$. So, at this stage, the system (2.2)-(2.4) is deduced but not equivalent to the initial one.*

Next, in order to specify the boundary conditions associated to (2.2)-(2.4), $\partial\Omega_2$ is divided into five parts as shown in Figure 1.2. We impose :

$$\left\{\begin{array}{l} \mathbf{G}\cdot\mathbf{n}=G_\Sigma \text{ on } \Sigma,\quad \mathbf{G}\cdot\mathbf{n}=0 \text{ on } \Gamma_2\cup\Gamma_3\cup\Gamma_4, \\ T=T_\Sigma \text{ on } \Sigma,\quad \mathbf{q}\cdot\mathbf{n}=0 \text{ on } \partial\Omega_2\setminus\Sigma, \\ \mathbf{u}\cdot\mathbf{n}=\frac{\mathbf{G}\cdot\mathbf{n}}{\rho} \text{ on } \Gamma_1,\quad \mathbf{u}\cdot\mathbf{n}=0 \text{ on } \Gamma_2\cup\Gamma_3\cup\Gamma_4, \\ \mathbf{u}\cdot\mathbf{t}=0 \text{ on } \Sigma,\quad \underline{\tau}\mathbf{n}\cdot\mathbf{t}=0 \text{ on } \partial\Omega_2\setminus\Sigma. \end{array}\right.$$

We still have to prescribe a boundary condition on Σ, which we take (in view of the coupling) of Neumann's type : $p-\underline{\tau}\mathbf{n}\cdot\mathbf{n}=p_\Sigma$.

Remark 2. *If one rather chooses to impose a Dirichlet condition $\mathbf{u}\cdot\mathbf{n}=\frac{G_\Sigma}{\rho}$ on Σ, then one can show (Cf. [6]) that the relation $div(r\rho\mathbf{u}) = div(r\mathbf{G})$ implies $\rho\mathbf{u}=\mathbf{G}$ in Ω_2, which justifies the proposed algorithm. In this case, the radial velocity is completely determined as $\frac{G_r}{\rho}$ and the corresponding momentum equation is just neglected. We prefer here to impose a Neumann condition on Σ and to use later the relation $\rho\mathbf{u}\cdot\mathbf{n}=\mathbf{G}\cdot\mathbf{n}$ as an additional transmission condition.*

A relevant issue concerns the boundary condition on the top of the wellbore. Let us notice that, even if the flowrate Q is known thanks to recorded data, one cannot impose it on the outflow boundary Γ_1 for the transport equation (2.2), since Q and G_Σ are related by the compatibility condition:

$$\int_{\Omega_2} r\frac{\rho-\rho^{n-1}}{\Delta t}dx+\int_{\Gamma_1} r\rho Q d\sigma+\int_\Sigma rG_\Sigma d\sigma=0.$$

Next, the 1.5D model is obtained as a conforming approximation of the 2D semi-discretized problem, by considering an explicit dependence of the unknowns on the radial coordinate. For the sake of simplicity, the velocity is taken here affine with respect to r whereas the scalar unknowns only depend on z :

$$\begin{array}{l} \mathbf{u}=\begin{pmatrix} u_r \\ u_z \end{pmatrix}=\begin{pmatrix} \frac{r}{R}\overline{u}_r(z) \\ \frac{r}{R}\overline{u}_z(z)+\frac{R-r}{R}\widehat{u_z}(z) \end{pmatrix}, \\ \mathbf{G}=\begin{pmatrix} G_r \\ G_z \end{pmatrix}=\begin{pmatrix} \frac{r}{R}\bar{G}_r(z) \\ G_z(z) \end{pmatrix},\quad \mathbf{q}=\begin{pmatrix} q_r \\ q_z \end{pmatrix}=\begin{pmatrix} \frac{r}{R}\bar{q}_r(z) \\ q_z(z) \end{pmatrix}, \\ \rho=\rho(z),\quad p=p(z),\quad T=T(z). \end{array} \tag{2.5}$$

Thanks to the boundary conditions, one further has $\overline{u}_r=0$ on Γ_2 and $\overline{u}_z=0$ on Σ.

The time-discretized problem is written under weak form, by means of a Petrov-Galerkin formulation for (2.2), respectively mixed variational formulations for (2.3) and (2.4). For this purpose, we introduce the following spaces :

$$\mathbf{W} = \Big\{ \mathbf{w} = \Big(\frac{r}{R}\ \bar{w}_r(z), w_z(z)\Big)^t;\ \bar{w}_r \in L^2(I),\ w_z \in H^1(I),$$
$$\mathbf{w}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \Big\} \subset H(div, \Omega_2),$$
$$\mathbf{V} = \Big\{ \mathbf{v} = \Big(\frac{r}{R}\ \bar{v}_r(z), v_z(r,z)\Big)^t;\ v_z = \frac{r}{R}\ \overline{v}_z(z) + \frac{R-r}{R}\ \widehat{v}_z(z),\ \bar{v}_r, \overline{v}_z, \widehat{v}_z \in H^1(I),$$
$$\mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4,\ \mathbf{v}\cdot\mathbf{t} = 0 \quad \text{on } \Sigma \Big\} \subset \mathbf{H}^1(\Omega_2),$$
$$\mathbf{H} = \{ \mathbf{w} \in \mathbf{W};\ \mathbf{w}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega_2 \setminus \Sigma \},$$
$$M = \{ q = q(z);\ q \in L^2(I) \} \subset L^2(\Omega_2)$$

as well as :

$$\mathbf{W}^* = \{ \mathbf{w} \in \mathbf{W};\ \mathbf{w}\cdot\mathbf{n} = G_\Sigma \text{ on } \Sigma \},\ \mathbf{V}^* = \{ \mathbf{v} \in \mathbf{V};\ \mathbf{v}\cdot\mathbf{n} = Q \text{ on } \Gamma_1 \}$$

where Q denotes here $\frac{\mathbf{G}\cdot\mathbf{n}}{\rho}$ and is assumed to be constant.

We consider the following weak formulations of problems (2.2), (2.3) and (2.4) :

$$\begin{cases} \text{Find } \mathbf{G} \in \mathbf{W}^* \\ \int_\Omega div(r\mathbf{G})\chi dx = -\int_\Omega r\frac{\rho - \rho^{n-1}}{\Delta t}\chi dx \quad \forall \chi \in M, \end{cases} \tag{2.6}$$

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V}^*,\ p \in M \\ m(\mathbf{u},\mathbf{v}) \quad + \quad n(p,\mathbf{v}) \quad = \quad l_1(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}^0 \\ n(q,\mathbf{u}) \quad = \quad l_2(q) \quad \forall q \in M, \end{cases} \tag{2.7}$$

$$\begin{cases} \text{Find } \mathbf{q} \in \mathbf{H},\ T \in M \\ a(\mathbf{q},\mathbf{w}) \quad + \quad b(T,\mathbf{w}) \quad = \quad f_1(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{H} \\ b(S,\mathbf{q}) \quad - \quad c(T,S) - \alpha d(T,S) \quad = \quad f_2(S) \quad \forall S \in M\ . \end{cases} \tag{2.8}$$

The bilinear forms are defined as follows :

$$\begin{aligned} & m(\mathbf{u},\mathbf{v}) = \textstyle\int_{\Omega_2} r\left(\frac{\rho}{\Delta t} + \kappa|\mathbf{G}|\right)\mathbf{u}\cdot\mathbf{v}dx + \int_{\Omega_2} r\mathbf{G}\cdot(v_r\nabla u_r + v_z\nabla u_z)dx \\ & \quad + \textstyle\int_{\Omega_2} \mu r(\partial_r u_z + \partial_z u_r)(\partial_r v_z + \partial_z v_r)dx \ + \ \frac{4}{3}\int_{\Omega_2} \mu r(\partial_z u_z - \frac{1}{R}\overline{u}_r)(\partial_z v_z - \frac{1}{R}\overline{v}_r)dx, \\ & n(q,\mathbf{v}) = -\textstyle\int_{\Omega_2} rq(\partial_z v_z + \frac{2}{R}\overline{v}_r)dx, \\ & a(\mathbf{q},\mathbf{w}) = \textstyle\int_{\Omega_2} \frac{r}{\lambda}\mathbf{q}\cdot\mathbf{w}dx, \quad b(S,\mathbf{w}) = \int_{\Omega_2} S div(r\mathbf{w})dx, \\ & c(T,S) = \textstyle\int_{\Omega_2} r\rho c_v\frac{TS}{\Delta t}dx, \quad d(T,s) = \int_{\Omega_2} r\ c_v\mathbf{G}\cdot\nabla T\ Sdx \end{aligned} \tag{2.9}$$

while the righthand-side terms are given by :

$l_1(\mathbf{v}) = \int_{\Omega_2} r\rho\mathbf{g}\cdot\mathbf{v}dx + \int_{\Omega_2} r\frac{\rho}{\Delta t}\mathbf{u}^{n-1}\cdot\mathbf{v}dx - \int_\Sigma Rp_\Sigma\mathbf{v}\cdot\mathbf{n}d\sigma,$

$l_2(q) = \int_{\Omega_2}\frac{r}{\rho^2}\left(\rho\frac{\rho-\rho^{n-1}}{\Delta t} + \mathbf{G}^n\cdot\nabla\rho\right)qdx,$

$f_1(\mathbf{w}) = \int_\Sigma RT_\Sigma\mathbf{w}\cdot\mathbf{n}d\sigma,$

$$f_2(S) = \int_{\Omega_2}\left(r\rho c_v\frac{T^{n-1}}{\Delta t} - \frac{r}{2}\left(\rho\frac{|\mathbf{u}|^2-|\mathbf{u}^{n-1}|^2}{\Delta t} + \mathbf{G}\cdot\nabla(|\mathbf{u}|^2)\right)\right. \\ \left. -div(r\,(p\underline{\mathbf{I}} - \underline{\tau})\,\mathbf{u}) + r\mathbf{g}\cdot\mathbf{G}S\right)dx.$$

It has been established in [2] that each of the previous problems has a unique solution when $\alpha = 0$, thanks to Babuška's theorem for (2.6), respectively to Babuška-Brezzi theorem for (2.7) and (2.8), under the assumption Δt sufficiently small. As to the energy balance (2.8) with convection (i.e. when $\alpha = 1$), its well-posedness is proved by using Fredholm's alternative, similarly to the reservoir case (Cf. [1]).

3. Coupling of Darcy-Forchheimer and Navier-Stokes Equations. We agree to denote by $\mathbf{n}$ the normal unit vector to Σ, oriented from the reservoir towards the wellbore. From now on, we shall index by 1 the unknowns related to the reservoir, respectively by 2 those related to the wellbore.

In this section, we introduce the transmission conditions which allow us to write the time-discretized coupled problem in mixed weak form and then we prove the uniqueness of the solution. The existence will be established in Section 4 by means of a Galerkin method based on the finite element spaces employed for the discretization.

In order to impose a flowrate Q at the wellbore head, and thus to take into account the recorded data, we turn to a global resolution of the coupled problem, at each time step. One thus overcomes the drawback of the sole wellbore problem.

3.1. Transmission conditions. The interface terms that have to be matched are those appearing by integration by parts in the 2D axisymmetric models, that is for the reservoir :

$$\int_\Sigma p_1\mathbf{G}_1'\cdot\mathbf{n}d\sigma - \int_\Sigma T_1\mathbf{q}_1'\cdot\mathbf{n}d\sigma,$$

respectively for the wellbore:

$$\int_\Sigma R(p_2 - \underline{\tau}_2\mathbf{n}\cdot\mathbf{n})\mathbf{u}_2'\cdot\mathbf{n}d\sigma - \int_\Sigma RT_2\mathbf{q}_2'\cdot\mathbf{n}d\sigma - \int_\Sigma R(\underline{\tau_2}\mathbf{n}\cdot\mathbf{t})\mathbf{u}_2'\cdot\mathbf{t}d\sigma.$$

When dealing with the coupling of Stokes and Darcy equations, one classically imposes the mass conservation and the balance of normal forces on the interface :

$$[\mathbf{G}\cdot\mathbf{n}] = 0, \quad [\underline{\sigma}\mathbf{n}\cdot\mathbf{n}] = 0, \tag{3.1}$$

where $[\cdot]$ stands for the jump across Σ and where the Cauchy tensors of the porous and the fluid media are respectively given by : $\underline{\sigma}_1 = -p_1\underline{\mathbf{I}}, \quad \underline{\sigma}_2 = -p_2\underline{\mathbf{I}} + \underline{\tau}_2$.

Due to the viscous context, one also has to prescribe a condition on the tangential component of the fluid's velocity. Several types of conditions exist in the literature. The one which seems to be in best agreement with experimental evidence is known as the Beavers-Joseph-Saffman law and it reads : $\mathbf{u}_2\cdot\mathbf{t} = -\frac{\sqrt{k}}{\delta}\underline{\sigma}_2\mathbf{n}\cdot\mathbf{t}$ with $\delta > 0$ a parameter experimentally determined and depending on many features of the interface (see [9] and references therein). However, the mathematical analysis doesn't lose in generality if one simply takes (as in [3] or [7]) $\mathbf{u}_2\cdot\mathbf{t} = 0$, since the previous condition only enhances the coercivity of the main operator. Indeed, one can notice that $\int_\Sigma R(\underline{\tau_2}\mathbf{n}\cdot\mathbf{t})\mathbf{u}_2'\cdot\mathbf{t}d\sigma$ is either null if we choose to impose $\mathbf{u}_2\cdot\mathbf{t}=0$ (and hence $\mathbf{u}_2'\cdot\mathbf{t}=0$) on Σ, or becomes an elliptic term if we choose the Beavers-Joseph-Saffman law. Therefore, in what follows we shall impose, for the sake of simplicity and in agreement with the wellbore model (cf. Section 2):

$$\mathbf{u}_2\cdot\mathbf{t}=0 \quad \text{on } \Sigma. \tag{3.2}$$

Next, the energetic aspect yields the continuity of the temperature and of the normal heat flux across Σ :

$$[T]=0, \quad [\mathbf{q}\cdot\mathbf{n}]=0. \tag{3.3}$$

Furthermore, we add the condition :

$$\rho_2\mathbf{u_2}\cdot\mathbf{n}=\mathbf{G_2}\cdot\mathbf{n} \tag{3.4}$$

which binds together the unknowns on Σ. So, the set of transmission conditions consists of (3.1) - (3.4).

3.2. Coupled problem in weak form. Similarly to Layton *et al.* [9] or to [7], we write a mixed weak formulation linking together the reservoir and the wellbore formulations.

According to Section 1, the reservoir model was written in the variational form (1.3), where we recall that the unknowns are denoted by $x_1=(\mathbf{G}_1,\mathbf{q}_1,p_1,T_1)$ and belong to the space X_1.

Concerning the wellbore, its unknowns are denoted by $x_2=(\mathbf{G}_2,\mathbf{u}_2,\mathbf{q}_2,p_2,T_2)$, its test-functions by $x_2'=(\chi,\mathbf{u}_2',\mathbf{q}_2',p_2',T_2')$ and belong respectively to :

$$X_2=\mathbf{W}\times\mathbf{V}\times\mathbf{H}\times M\times M, \quad Y_2=M\times\mathbf{V}^0\times\mathbf{H}\times M\times M.$$

It is useful to introduce the affine set : $X_2^*=\mathbf{W}^*\times\mathbf{V}^*\times\mathbf{H}\times M\times M.$

We recall that the wellbore model is nonlinear. In order to simplify the presentation, we choose to replace at each t^n, $\mathbf{G}_2$ by $\mathbf{G}_2^{n-1}$ in the momentum and the energy equations. This allows us to write the global 1.5D wellbore problem as follows :

$$\begin{cases} \text{Find } x_2\in X_2^* \\ \mathcal{A}_2(x_2,x_2')=\mathcal{F}_2(x_2'),\ \forall x_2'\in Y_2. \end{cases} \tag{3.5}$$

Remark 3. *One doesn't lose in generality due to the latter linearization with respect to* $\mathbf{G}_2$. *Indeed, thanks to the decoupling of the wellbore equations, the well-posedness of the nonlinear problem only requires the invertibility of the operator* $\mathcal{A}_2$.

Next, in order to obtain the mixed formulation of the coupled problem, we dualize the transmission conditions on Σ by means of Lagrange multipliers. For this purpose, let us first introduce the following spaces :

$$\begin{aligned}
\mathbb{X}&=\{x=(x_1,x_2)\in X_1\times X_2\,;\ \mathbf{G}_1\cdot\mathbf{n},\,\mathbf{q}_1\cdot\mathbf{n}\in L^2(\Sigma)\},\\
\mathbb{Y}&=\{x'=(x_1',x_2')\in X_1\times Y_2\,;\ \mathbf{G}_1\cdot\mathbf{n},\,\mathbf{q}_1\cdot\mathbf{n}\in L^2(\Sigma)\},\\
\mathbb{Y}^0&=\{x'\in\mathbb{Y}\,;\ \mathbf{G}_1'\cdot\mathbf{n}=0 \text{ on } \Upsilon_{\mathbf{G}}\setminus\Sigma,\ \mathbf{q}_1'\cdot\mathbf{n}=0 \text{ on } \Upsilon_{\mathbf{q}}\setminus\Sigma,\ \mathbf{u}_2'\cdot\mathbf{n}=0 \text{ on } \Gamma_1\},\\
\mathbb{X}^*&=\{x\in\mathbb{X}\,;\ \mathbf{G}_1\cdot\mathbf{n}=0 \text{ on } \Upsilon_{\mathbf{G}}\setminus\Sigma,\ \mathbf{q}_1\cdot\mathbf{n}=q^* \text{ on } \Upsilon_{\mathbf{q}}\setminus\Sigma,\ \mathbf{u}_2\cdot\mathbf{n}=Q \text{ on } \Gamma_1\}.
\end{aligned}$$

The Hilbert spaces $\mathbb{X}$ and $\mathbb{Y}$ are endowed with the graph norms.

REMARK 4. *The previous spaces are obtained by removing the boundary conditions on Σ from the formulations (1.3) and (3.5), and by adding some more regularity on the normal traces of $\mathbf{G}_1$, $\mathbf{q}_1$ on Σ.*

We also introduce the multipliers' spaces :

$$\mathbb{L} = (L^2(\Sigma))^2, \quad \mathbb{K} = (L^2(\Sigma))^3$$

and the bilinear forms on $\mathbb{L} \times \mathbb{Y}$, respectively $\mathbb{K} \times \mathbb{X}$:

$$\mathcal{I}(\Lambda, x') = \int_\Sigma (\mathbf{G}_1' \cdot \mathbf{n} - R\mathbf{u}_2' \cdot \mathbf{n})\theta d\sigma - \int_\Sigma (\mathbf{q}_1' \cdot \mathbf{n} - R\mathbf{q}_2' \cdot \mathbf{n})\mu d\sigma,$$

$$\mathcal{J}(\Lambda', x) = \int_\Sigma (\mathbf{G}_1 \cdot \mathbf{n} - R\rho_2\mathbf{u}_2 \cdot \mathbf{n})\theta' d\sigma + \int_\Sigma (\mathbf{G}_1 \cdot \mathbf{n} - R\mathbf{G}_2 \cdot \mathbf{n})\zeta' d\sigma - \int_\Sigma (\mathbf{q}_1 \cdot \mathbf{n} - R\mathbf{q}_2 \cdot \mathbf{n})\mu' d\sigma$$

for any $x \in \mathbb{X}$, $x' \in \mathbb{Y}$, $\Lambda = (\theta, \mu) \in \mathbb{L}$, $\Lambda' = (\zeta', \theta', \mu') \in \mathbb{K}$. Then, putting

$$\mathcal{A}(x, x') = \mathcal{A}_1(x_1, x_1') + \mathcal{A}_2(x_2, x_2'), \quad \forall x \in \mathbb{X}, \forall x' \in \mathbb{Y},$$

$$\mathcal{F}(x') = \mathcal{F}_1(x_1') + \mathcal{F}_2(x_2'), \quad \forall x' \in \mathbb{Y},$$

the coupled problem can be written as follows :

$$\begin{cases} \text{Find } x \in \mathbb{X}^*, \Lambda \in \mathbb{L} \\ \mathcal{A}(x, x') \quad + \quad \mathcal{I}(\Lambda, x') \quad = \quad \mathcal{F}(x'),\ \forall x' \in \mathbb{Y}^0 \\ \mathcal{J}(\Lambda', x) \quad = \quad 0,\ \forall \Lambda' \in \mathbb{K}. \end{cases} \tag{3.6}$$

REMARK 5. *The bilinear form $\mathcal{J}(\cdot,\cdot)$ dualizes the boundary condition for the radial velocity as well as the continuity of the normal specific and heat fluxes across the interface. Meanwhile, the bilinear form $\mathcal{I}(\cdot,\cdot)$ takes into account the interface terms appearing in the two reservoir and wellbore problems after integration by parts. The multiplier $\Lambda = (\theta, \mu)$ can be interpreted as (p_1, T_1), or still as $(p_2 - \underline{\tau}_2\mathbf{n} \cdot \mathbf{n}, T_2)$.*

3.3. Uniqueness of the solution. This subsection is devoted to the mathematical analysis of the mixed formulation (3.6). For the sake of clarity, let us briefly present the roadmap. We first establish (in Lemma 3.1) that $\mathcal{I}$ and $\mathcal{J}$ satisfy both an inf-sup condition, therefore it is sufficient to study the following problem :

$$\begin{cases} \text{Find } x \in \mathbb{J}^* \\ \mathcal{A}(x, x') = \mathcal{F}(x'),\ \forall x' \in \mathbb{I} \end{cases} \tag{3.7}$$

where :

$$\mathbb{J}^* = \{x \in \mathbb{X}^* \,;\, \mathcal{J}(\Lambda', x) = 0,\ \forall \Lambda' \in \mathbb{K}\}, \quad \mathbb{I} = \left\{x' \in \mathbb{Y}^0 \,;\, \mathcal{I}(\Lambda, x') = 0,\ \forall \Lambda \in \mathbb{L}\right\}.$$

Indeed, thanks to the general theory of saddle point problems (Cf. for instance [5]), one then knows that for any x solution of (3.7), there exists a unique multiplier $\Lambda \in \mathbb{L}$ such that the pair (x, Λ) satisfies the mixed problem (3.6).

We next prove uniqueness of the solution for (3.7) in Theorem 3.5, by means of the classical Babuška theorem. However, since the operator $\mathcal{A}$ is non-standard, we couldn't establish a second inf-sup condition for $\mathcal{A}$, ensuring the existence.

LEMMA 3.1. *The following two inf-sup conditions hold :*

$$\exists b_1 > 0,\ \forall \Lambda \in \mathbb{L}, \quad \sup_{x' \in \mathbb{Y}^0} \frac{\mathcal{I}(\Lambda, x')}{\|x'\|_{\mathbb{Y}}} \geq b_1 \|\Lambda\|_{0,\Sigma}\,, \tag{3.8}$$

$$\exists b_2 > 0,\ \forall \Lambda' \in \mathbb{K}, \quad \sup_{x \in \mathbb{X}^0} \frac{\mathcal{J}(\Lambda', x)}{\|x\|_{\mathbb{X}}} \geq b_2 \|\Lambda'\|_{0,\Sigma}\,. \tag{3.9}$$

Proof. We make use of Fortin's trick. In order to establish (3.8), with any $\Lambda = (\theta, \mu) \in \mathbb{L}$ we associate $x' \in \mathbb{Y}^0$ satisfying :

$$\|x'\|_{\mathbb{Y}} \leq c \|\Lambda\|_{0,\Sigma}\,, \quad \mathcal{I}(\Lambda, x') = \|\Lambda\|^2_{0,\Sigma}\,.$$

We consider $x' \in \mathbb{Y}^0$ such that all its components are null, except for $\mathbf{G}'_1$ and $\mathbf{q}'_2$ which are taken as follows : $\mathbf{q}'_2 = \begin{pmatrix} \frac{r}{R^2}\tilde{\mu} \\ 0 \end{pmatrix}$ and $\mathbf{G}'_1 = \nabla\varphi$, where $\tilde{\mu}$ is the extension of μ by 0 on Γ_2 and φ is the unique solution of the auxiliary boundary value problem :

$$\begin{cases} \Delta\varphi = f & \text{in } \Omega_1 \\ \frac{\partial\varphi}{\partial n} = g & \text{on } \Sigma \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } \Upsilon_G \setminus \Sigma \\ \varphi = 0 & \text{on } \Upsilon_p, \end{cases} \tag{3.10}$$

with data $(f, g) = (0, \theta)$. It is well-known that $|\varphi|_{1,\Omega_1} \leq c \|\theta\|_{0,\Sigma}$ with c only depending on the domain. It is then obvious that $\mathbf{q}'_2$ and $\mathbf{G}'_1$ thus defined belong to $\mathbf{H}$, respectively $H(div, \Omega_1)$ and satisfy the boundary conditions :

$$\mathbf{q}'_2 \cdot \mathbf{n} = 0 \ \text{ on } \partial\Omega_2 \setminus \Sigma, \qquad \mathbf{G}'_1 \cdot \mathbf{n} = 0 \ \text{ on } \Upsilon_{\mathbf{G}} \setminus \Sigma$$

together with the estimate :

$$\|\mathbf{q'}_2\|_{H(div,\Omega_2)} + \|\mathbf{G'}_1\|_{H(div,\Omega_1)} + \|\mathbf{q'}_2 \cdot \mathbf{n}\|_{0,\Sigma} + \|\mathbf{G'}_1 \cdot \mathbf{n}\|_{0,\Sigma} \leq c(\|\mu\|_{0,\Sigma} + \|\theta\|_{0,\Sigma}).$$

Therefore, condition (3.8) holds. We use a similar idea in order to prove (3.9). With any $\Lambda' = (\zeta', \theta', \mu') \in \mathbb{K}$, we associate $x \in \mathbb{X}^0$ whose components are null except for $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{q}_2$. This already yields :

$$\mathcal{J}(\Lambda', x) = \int_\Sigma \mathbf{G}_1 \cdot \mathbf{n}\theta' d\sigma + \int_\Sigma (\mathbf{G}_1 \cdot \mathbf{n} - R\mathbf{G}_2 \cdot \mathbf{n})\zeta' d\sigma + \int_\Sigma \mathbf{q}_2 \cdot \mathbf{n}\mu' d\sigma.$$

Then we construct $\mathbf{G}_1$ and $\mathbf{q}_2$ as above, while for $\mathbf{G}_2$ we take $\begin{pmatrix} \frac{r}{R^2}(\theta' - \zeta') \\ 0 \end{pmatrix}$. It follows that $\|x\|_{\mathbb{X}} \leq c \|\Lambda'\|_{0,\Sigma}$ and $\mathcal{J}(\Lambda', x) = \|\Lambda\|^2_{0,\Sigma}$, which ends the proof. ∎

Therefore, in what follows we study the problem (3.7). By separating the vector functions from the scalar ones and by consequently putting :

$$\mathbb{J}^* = \mathbb{U}^* \times \mathbb{S}, \quad \mathbb{I} = \mathbb{T} \times \mathbb{S},$$

one can still write (3.7) as follows :

$$\begin{cases} \text{Find } (\mathbf{U}, s) \in \mathbb{U}^* \times \mathbb{S} \\ \mathtt{A}(\mathbf{U}, \mathbf{U}') + \mathtt{B}(s, \mathbf{U}') = \mathtt{F}_1(\mathbf{U}'), \qquad \forall \mathbf{U}' \in \mathbb{T}^0 \\ \mathtt{B}(s', \mathbf{U}) - \mathtt{C}(s, s') - \alpha \mathtt{D}(s, s') = \mathtt{F}_2(s'), \quad \forall s' \in \mathbb{S} \end{cases} \tag{3.11}$$

where $\mathbf{U}$ represents $(\mathbf{G}_1, \mathbf{q}_1, \mathbf{G}_2, \mathbf{u}_2, \mathbf{q}_2)$, the corresponding test-function $\mathbf{U}'$ stands for $(\mathbf{G}'_1, \mathbf{q}'_1, \chi, \mathbf{u}'_2, \mathbf{q}'_2)$, whereas $s = (p_1, T_1, p_2, T_2)$. Here above, we have put :

$$\mathtt{A}(\mathbf{U}, \mathbf{U}') = \int_{\Omega_1} \tfrac{1}{r} M \mathbf{G}_1 \cdot \mathbf{G}'_1 dx + \int_{\Omega_1} \tfrac{1}{r\lambda_1} \mathbf{q}_1 \cdot \mathbf{q}'_1 dx$$

$$+ \int_{\Omega_2} \chi div(r\mathbf{G}_2) dx + \int_{\Omega_2} \tfrac{r}{\lambda_1} \mathbf{q}_2 \cdot \mathbf{q}'_2 dx + m(\mathbf{u}_2, \mathbf{u}'_2),$$

$$\mathtt{B}(s, \mathbf{U}') = -\int_{\Omega_1} p_1 div \mathbf{G}'_1 dx + \int_{\Omega_1} T_1 div \mathbf{q}'_1 dx - \int_{\Omega_2} p_2 div(r\mathbf{u}'_2) dx + \int_{\Omega_2} T_2 div(r\mathbf{q}'_2) dx,$$

$$\mathtt{C}(s, s') = \int_{\Omega_1} r \tfrac{a}{\Delta t} p_1 p'_1 dx - \int_{\Omega_1} r \tfrac{b}{\Delta t} T_1 p'_1 dx$$

$$+ \int_{\Omega_1} r \tfrac{d}{\Delta t} T_1 T'_1 dx - \int_{\Omega_1} r \tfrac{f}{\Delta t} p_1 T'_1 dx + \int_{\Omega_2} r \tfrac{c_v \rho_2}{\Delta t} T_2 T'_2 dx,$$

$$\mathtt{D}(s, s') = \int_{\Omega_1} \kappa \mathbf{G}_1^{n-1} \cdot \nabla T_1 T'_1 dx + \int_{\Omega_1} l \mathbf{G}_1^{n-1} \cdot \nabla p_1 T'_1 dx + \int_{\Omega_2} r\ c_v \mathbf{G}_2^{n-1} \cdot \nabla T_2\ S_2 dx.$$

We refer to (2.9) for the definition of $m(\cdot, \cdot)$. Let us note that neither $\mathtt{A}(\cdot, \cdot)$ nor $\mathtt{C}(\cdot, \cdot)$ are symmetric and moreover, the spaces employed for the solution and the test-functions are different. Hence, one cannot apply the existing generalizations of the Babuška-Brezzi theorem (Cf. [5], [13] or [17]) in the case $\alpha = 0$. We next establish some preliminary results (Lemmas 3.2, 3.3 and 3.4) which will finally allow us to prove that the operator $\mathcal{A}_0 = \begin{bmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{B}^T & -\mathtt{C} \end{bmatrix}$ is injective.

Lemma 3.2. *There exist two constants β_1 and β_2 independent of Δt such that :*

$$\forall s \in \mathbb{S}, \quad \sup_{\mathbf{U} \in \mathbb{U}^0} \frac{\mathtt{B}(s, \mathbf{U})}{\|\mathbf{U}\|} \geq \beta_1 \|s\| \quad \textit{and} \quad \sup_{\mathbf{U}' \in \mathbb{T}^0} \frac{\mathtt{B}(s, \mathbf{U}')}{\|\mathbf{U}'\|} \geq \beta_2 \|s\|.$$

Proof. We follow the proofs of the inf-sup conditions related to the wellbore and the reservoir models.

In order to establish the first relation, with any given $s = (p_1, T_1, p_2, T_2) \in \mathbb{S}$ we associate a function $\mathbf{U} \in \mathbb{U}^0$ satisfying $\mathtt{B}(s, \mathbf{U}) \geq c_1 \|s\|^2$ and $\|\mathbf{U}\| \leq c_2 \|s\|$. For this purpose, we take $\overline{u}_z = 0$, $\overline{u}_r \in H_0^1(\Sigma)$ satisfying

$$\int_\Sigma \overline{u}_r dz = \frac{R}{6} \int_I p_2 dz, \qquad \|\overline{u}_r\|_{1,\Sigma} \leq c\, \|p_2\|_{0,I}$$

and $\widehat{u}_z(\zeta) = \int_{z_{\min}}^{\zeta} (p_2 + \frac{6}{R} \overline{u}_r) dz$. Then we put $\mathbf{u}_2 = \begin{pmatrix} \frac{r}{R} \overline{u}_r \\ \frac{r}{R} \overline{u}_z + \frac{R-r}{R} \hat{u}_z \end{pmatrix}$ and we get :

$$\int_{\Omega_2} p_2 div(r\mathbf{u}_2) dx = \tfrac{R^2}{6} \int_I p_2 (2\partial_z \overline{u}_z + \partial_z \hat{u}_z + \tfrac{6}{R} \overline{u}_r) dz = \tfrac{R}{6} \|p_2\|^2_{0,\Omega_2},$$

$$\|\mathbf{u}_2\|_{1,\Omega_2} \leq c\, \|p_2\|_{0,\Omega_2}.$$

Next, we consider $\mathbf{G}_2 = \begin{pmatrix} \frac{r}{R}\rho_2\overline{u}_r \\ 0 \end{pmatrix}$ and $\mathbf{G}_1 = \nabla\xi$, where ξ is the solution of (3.10) with data $(f,g) = (-p_1, R\rho_2\overline{u}_r)$. Since $\|\xi\|_{1,\Omega_1} \leq c(\|p_1\|_{0,\Omega_1} + \|\overline{u}_r\|_{0,\Sigma})$, one clearly has :

$$\|\mathbf{G}_2\|_{H(div,\Omega_2)} + \|\mathbf{G}_1\|_{H(div,\Omega_1)} + \|\mathbf{G}_1 \cdot \mathbf{n}\|_{0,\Sigma} \leq c(\|p_1\|_{0,\Omega_1} + \|p_2\|_{0,\Omega_2}).$$

We proceed similarly for $\mathbf{q}_1$ and $\mathbf{q}_2$. More precisely, we choose $\mathbf{q}_2 = \begin{pmatrix} \frac{r}{R}\overline{q}_{2r}(z) \\ q_{2z}(z) \end{pmatrix}$ associated with T_2 exactly as in the sole wellbore problem, that is $\mathbf{q}_2$ is defined by :

$$\begin{cases} \overline{q}_{2r} = 0 \text{ on } \Gamma_2, \quad \overline{q}_{2r} = \frac{R}{2m(\Sigma)}\int_I T_2 dz \text{ on } \Sigma, \\ q_{2z}(\zeta) = \int_{z_{\min}}^{\zeta} \left(T_2 - \frac{2}{R}\overline{q}_{2r}\right) dz. \end{cases}$$

Then obviously $\|\mathbf{q}_2\|_{H(div,\Omega_2)} \leq c\|T_2\|_{0,\Omega_2}$ and

$$\int_{\Omega_2} T_2 div(r\mathbf{q}_2)dx = \frac{R^2}{2}\int_I T_2 \left(\partial_z q_{2z} + \frac{2}{R}\overline{q}_{2r}\right) dz = \frac{R}{2}\|T_2\|^2_{0,\Omega_2}.$$

Finally, we put $\mathbf{q}_1 = \nabla\zeta$, where ζ satisfies (3.10) with data $(f,g) = (T_1, R\mathbf{q}_2 \cdot \mathbf{n})$. The above choice for $\mathbf{U}$ implies that the transmission conditions on Σ are checked :

$$\mathbf{G}_1 \cdot \mathbf{n} = R\mathbf{G}_2 \cdot \mathbf{n} = R\rho_2\overline{u}_r, \quad \mathbf{q}_1 \cdot \mathbf{n} = R\mathbf{q}_2 \cdot \mathbf{n}$$

and yields the desired condition.

The proof of the second inf-sup condition is quite similar : one simply chooses $\chi = 0$, $\mathbf{u}'_2$, $\mathbf{q}'_2$ and $\mathbf{q}'_1$ as above, while for $\mathbf{G}'_1$ we now substitute the boundary condition on Σ by $\frac{\partial\xi}{\partial n} = R\overline{u}_r$. ∎

LEMMA 3.3. *There exists a positive constant γ, depending on Δt, such that :*

$$\forall s \in \mathbb{S}, \quad \mathtt{C}(s,s) \geq \gamma(\|p_1\|^2_{0,\Omega_1} + \|T_1\|^2_{0,\Omega_1} + \|T_2\|^2_{0,\Omega_2}).$$

Proof. It follows from the study of the reservoir model, see [1]. Note that $\mathtt{C}$ is not positive definite, since the norm of p_2 is missing from the previous estimate. ∎

LEMMA 3.4. *For Δt sufficiently small, the following statement holds :*

$$\forall \mathbf{U} \in \mathbb{U}^0 \setminus \{0\}, \qquad \sup_{\substack{\mathbf{U}' \in \mathbb{T}^0 \\ \mathbf{U} - \mathbf{U}' \in Ker\mathtt{B}}} \frac{\mathtt{A}(\mathbf{U},\mathbf{U}')}{\|\mathbf{U}\|\|\mathbf{U}'\|} > 0.$$

Proof. In order to establish the result, it suffices to construct a linear continuous operator $\mathcal{R} : \mathbb{U}^0 \to \mathbb{T}^0$ satisfying :

$$\begin{array}{c} \mathtt{B}(s,\mathbf{U}) = \mathtt{B}(s,\mathcal{R}\mathbf{U}), \quad \forall s \in \mathbb{S}, \\ \mathtt{A}(\mathbf{U},\mathcal{R}\mathbf{U}) > 0, \quad \forall \mathbb{U}^0 \setminus \{0\}. \end{array}$$

So let $\mathbf{U} = (\mathbf{G}_1, \mathbf{q}_1, \mathbf{G}_2, \mathbf{u}_2, \mathbf{q}_2) \in \mathbb{U}^0$ satisfying : $\mathbf{G}_1 \cdot \mathbf{n} = R\mathbf{G}_2 \cdot \mathbf{n} = R\rho_2\mathbf{u}_2 \cdot \mathbf{n}$, $\mathbf{q}_1 \cdot \mathbf{n} = R\mathbf{q}_2 \cdot \mathbf{n}$ on Σ. Then we take $\mathcal{R}\mathbf{U} = \mathbf{U}' = (\mathbf{G}'_1, \mathbf{q}_1, \chi, \mathbf{u}_2, \mathbf{q}_2)$ where $\mathbf{G}'_1$ and χ will be defined later such that :

$$\mathbf{G}'_1 \cdot \mathbf{n} = \frac{1}{\rho_2}\mathbf{G}_1 \cdot \mathbf{n} \text{ on } \Sigma, \quad div\mathbf{G}'_1 = div\mathbf{G}_1 \text{ in } \Omega_1, \quad \|\mathbf{G}'_1\|_{0,\Omega_1} + \|\chi\|_{0,\Omega_2} \leq c\,\|\mathbf{U}\|.$$

Then obviously $\mathbf{U}'$ belongs to $\mathbb{T}^0$, satisfies $\|\mathbf{U}'\| \le c\|\mathbf{U}\|$ and $\mathbf{U}-\mathbf{U}' \in Ker\mathtt{B}$. Moreover, one has :

$$\mathtt{A}(\mathbf{U},\mathbf{U}') \ge c\left(\|\mathbf{q}_1\|^2_{0,\Omega_1} + \|\mathbf{q}_2\|^2_{0,\Omega_2}\right) + \int_{\Omega_1} \frac{1}{r}\underline{\mathbf{M}}\mathbf{G}_1\cdot\mathbf{G}'_1 dx + \int_{\Omega_2} \chi div(r\mathbf{G}_2)dx + m(\mathbf{u}_2,\mathbf{u}_2)$$

where according to (2.9),

$$\begin{aligned} m(\mathbf{u}_2,\mathbf{u}_2) = &\int_{\Omega_2} r\left(\frac{\rho}{\Delta t} + \kappa|\mathbf{G}_2|\right)|\mathbf{u}_2|^2 dx + \int_{\Omega_2} r\mathbf{G}_2\cdot\mathbf{u}_2\nabla\mathbf{u}_2 dx + \\ &+ \int_{\Omega_2} \mu r(\partial_r u_z + \partial_z u_r)^2 dx \ + \ \frac{4}{3}\int_{\Omega_2} \mu r(\partial_z u_z - \frac{1}{R}\overline{u}_r)^2 dx. \end{aligned}$$

Using the dependence on r of $\mathbf{u}_2$, one gets after integrating with respect to r (see also [2] for more details) :

$$\begin{aligned} \int_{\Omega_2} r(\partial_r u_z + \partial_z u_r)^2 dx &= \int_I \left(\frac{1}{2}(\overline{u}_z - \hat{u}_z)^2 + \frac{R^2}{4}(\partial_z\overline{u}_r)^2 + \frac{2R}{3}\partial_z\overline{u}_r(\overline{u}_z - \hat{u}_z)\right) dz \\ &\ge c(\int_{\Omega_2} r\left(\partial_r u_z\right)^2 dx + \int_{\Omega_2} r\left(\partial_z u_r\right)^2 dx) \end{aligned}$$

with c a numeric constant, while the mean's inequality implies that :

$$\int_{\Omega_2} r(\partial_z u_z - \frac{1}{R}\overline{u}_r)^2 dx \ge \frac{1}{2}\int_{\Omega_2} r\left(\partial_z u_z\right)^2 dx - \frac{2}{R^2}\int_{\Omega_2} r u_r^2 dx.$$

Furthermore, bounding the convective term by means of Young's inequality yields :

$$\begin{aligned} m(\mathbf{u}_2,\mathbf{u}_2) \ge\ & c(1-\epsilon)\mu\left(\int_{\Omega_2} r\left(\partial_z u_z\right)^2 dx + \int_{\Omega_2} r\left(\partial_r u_z\right)^2 dx + \int_{\Omega_2} r\left(\partial_z u_r\right)^2 dx\right) \\ &+ \int_{\Omega_2} r\left(\frac{\rho}{\Delta t} + \kappa|\mathbf{G}_2| - \frac{|\mathbf{G}_2|^2}{4\mu\epsilon c}\right) u_z^2 dx \\ &+ \int_{\Omega_2} r\left(\frac{\rho}{\Delta t} + \kappa|\mathbf{G}_2| + \frac{5}{4R}G_r - \frac{8\mu}{3R^2} - \frac{G_z^2}{4\mu\epsilon c}\right) u_r^2 dx, \end{aligned}$$

for any $\epsilon \in]0,1[$.

Let us now construct $\mathbf{G}'_1$. For this purpose, we consider the problem (3.10) with $(f,g) = (0, R(1-\rho_2)\overline{u}_r)$. Its unique solution $\psi \in H^1(\Omega_1)$ satisfies : $|\psi|_{1,\Omega_1} \le K\|u_r\|_{0,\Omega_2}$, where the constant K only depends on the domain Ω_1, on the density ρ_2 and on the well's radius R. So one can now put $\mathbf{G}'_1 = \nabla\psi + \mathbf{G}_1$ and obtain, on the one hand :

$$\|\mathbf{G}'_1\|_{0,\Omega_1} \le c\left(\|\mathbf{G}_1\|_{0,\Omega_1} + \|u_r\|_{0,\Omega_2}\right).$$

On the other hand, Young's inequality implies that:

$$\begin{aligned} \int_{\Omega_1} \frac{1}{r}\underline{\mathbf{M}}\mathbf{G}_1\cdot\mathbf{G}'_1 dx &= \int_{\Omega_1} \frac{1}{r}\underline{\mathbf{M}}\mathbf{G}_1\cdot\mathbf{G}_1 dx + \int_{\Omega_1} \frac{1}{r}\underline{\mathbf{M}}\mathbf{G}_1\cdot\nabla\psi dx \\ &\ge a(1-\delta)\|\mathbf{G}_1\|^2_{0,\Omega_1} - \frac{1}{4\delta}|\psi|^2_{1,\Omega_1} \end{aligned}$$

where a is the coercivity constant of the positive definite tensor $\frac{1}{r}\underline{\mathbf{M}}$ and $\delta \in]0,1[$ is an arbitrary parameter.

One still has to choose χ. For this purpose, let us notice that :

$$\int_{\Omega_2} \chi div(r\mathbf{G}_2)dx = \frac{R^2}{2}\int_I \chi(\partial_z G_{2z} + \frac{2}{R}\overline{G}_{2r})dz,$$

since $\mathbf{G}_2 = \begin{pmatrix} \frac{r}{R}\overline{G}_{2r}(z) \\ G_{2z}(z) \end{pmatrix}$. Then by putting $\chi = \upsilon(\partial_z G_{2z} - \frac{2}{R}\overline{G}_{2r})$ with υ a positive constant and by using that $\overline{G}_{2r} = \rho_2 \overline{u}_r$ on Σ, one gets :

$$\|\chi\|_{0,\Omega_2} \leq c\,\|\mathbf{G}_2\|_{H(div,\Omega_2)}\,,$$
$$\int_{\Omega_2} \chi div(r\mathbf{G}_2)dx \geq \frac{\upsilon R}{2}\left(\|\partial_z G_{2z}\|^2_{0,\Omega_2} - \frac{12\overline{\rho}^2}{R^2}\|u_r\|^2_{0,\Omega_2}\right) \geq c\,\|G_{2z}\|^2_{1,\Omega_2} - \frac{6\upsilon\overline{\rho}^2}{R}\|u_r\|^2_{0,\Omega_2}\,,$$

thanks to Friedrichs-Poincaré's inequality for G_{2z}.

Using that $\int_{\Omega_2} u_r^2 dx = \frac{4}{3R}\int_{\Omega_2} r u_r^2 dx$, it is next possible to choose Δt, as well as the parameters $\epsilon, \delta \in]0,1[$ and $\upsilon > 0$, such that :

$$\frac{\rho}{\Delta t} > \max\left(\frac{|\mathbf{G}_2|^2}{4\mu\epsilon c} - \kappa|\mathbf{G}_2|,\ \frac{8\mu}{3R^2} + \frac{G_z^2}{4\mu\epsilon c} + \frac{K^2}{3R\delta} + \frac{8\upsilon\overline{\rho}^2}{R^2} - \kappa|\mathbf{G}_2| - \frac{5}{4R}G_r\right).$$

It follows that there exists $\alpha > 0$ such that :

$$\mathtt{A}(\mathbf{U},\mathbf{U}') \geq \alpha\left(\|\mathbf{G}_1\|^2_{0,\Omega_1} + \|\mathbf{q}_1\|^2_{0,\Omega_1} + \|\mathbf{u}_2\|^2_{1,\Omega_2} + \|\mathbf{q}_2\|^2_{0,\Omega_2} + \|\mathbf{G}_2\|^2_{H(div,\Omega_2)}\right) \quad (3.12)$$

so the Lemma is established. $\square$

THEOREM 3.5. *For Δt sufficiently small, the following statement is true :*

$$\forall x \in \mathbb{J}^0 \setminus \{0\}\,, \quad \sup_{x' \in \mathbb{I}} \frac{\mathcal{A}_0(x,x')}{\|x'\|_{\mathbb{Y}}} > 0. \quad (3.13)$$

Therefore, problems (3.7) and (3.6) have at most one solution for $\alpha = 0$.

Proof. We focus on problem (3.7). It is sufficient to prove that the homogeneous problem admits only the trivial solution. So, let $(\mathbf{U}, s) \in \mathbb{U}^0 \times \mathbb{S}$ satisfy :

$$\begin{cases} \mathtt{A}(\mathbf{U},\mathbf{U}') + \mathtt{B}(s,\mathbf{U}') = 0, & \forall \mathbf{U}' \in \mathbb{T}^0 \\ -\mathtt{B}(s',\mathbf{U}) + \mathtt{C}(s,s') = 0, & \forall s' \in \mathbb{S} \end{cases}$$

and let us take $s' = s$ and $\mathbf{U}' = \mathcal{R}\mathbf{U}$, where $\mathcal{R}$ is the operator introduced in Lemma 3.4. Then, by adding the above equations and by using the positivity of $\mathtt{A}(\cdot,\cdot)$ and $\mathtt{C}(\cdot,\cdot)$, according to Lemmas 3.3 and 3.4, it follows that $\mathbf{U} = \mathbf{0}$ and also $(p_1, T_1, T_2) = \mathbf{0}$. One still has to show that p_2 is also null. For this purpose, one uses the second inf-sup condition established in Lemma 3.2 :

$$\beta_2\|s\| \leq \sup_{\mathbf{U}' \in \mathbb{T}^0} \frac{\mathtt{B}(s,\mathbf{U}')}{\|\mathbf{U}'\|} = \sup_{\mathbf{U}' \in \mathbb{T}^0} \frac{\mathtt{A}(\mathbf{U},\mathbf{U}')}{\|\mathbf{U}'\|} = 0$$

which ends the proof. The uniqueness of the solution of the mixed problem (3.6) holds thanks to Lemma 3.1. $\square$

REMARK 6. *One may note that the L^2-norms of the terms $div\mathbf{G}_1$, $div\mathbf{q}_1$ and $div\mathbf{q}_2$ are missing from the estimate (3.12). At this stage, we couldn't establish the second inf-sup condition for $\mathcal{A}_0$:*

$$\exists c > 0, \quad \forall x' \in \mathbb{I}, \quad \sup_{x \in \mathbb{J}^0} \frac{\mathcal{A}_0(x,x')}{\|x\|_{\mathbb{X}}} \geq c\|x'\|_{\mathbb{Y}}.$$

Therefore, we couldn't apply Babuška's theorem in order to get the existence, too. This will be proved in the next section.

4. Discrete coupled problem.

4.1. Finite element approximation . Let $(\mathcal{T}_h^1)_{h>0}$ be a regular family of triangulations of $\bar{\Omega}_1$ consisting of triangles and $(\mathcal{T}_h^2)_{h>0}$ a family of triangulations of $\bar{\Omega}_2$ consisting of rectangles, with only one cell in the radial direction. In what follows, we suppose that the two meshes are matching on the perforations Σ and we agree to denote by $\mathcal{E}_h$ the set of edges situated on Σ. We shall use the notation $h_{min,\Sigma} = \min_{e\in\mathcal{E}_h} h_e$. We also assume that :

$$(\mathbf{H}) \quad \overline{\rho} \geq \rho_{2h}(z) \geq \underline{\rho} > 0 \quad \text{a.e. on } \Sigma$$

where ρ_{2h} is a piecewise constant approximation of ρ_2 on $\mathcal{T}_h^2$. We next write a conforming approximation of problem (3.6) based on the finite element spaces already used for the separate reservoir and wellbore models, that is : the lowest-order Raviart-Thomas elements for the fluxes $\mathbf{G}$ and $\mathbf{q}$, P_0 elements for the pressure, the temperature and implicitly the density and $(Q_1)^2$-continuous elements for the fluid's velocity $\mathbf{u}_2$. It is useful to introduce the finite dimensional spaces :

$$M_h = \left\{q \in L^2(\Omega_1);\ q_{/T} \in P_0,\ \forall T \in \mathcal{T}_h^1\right\},$$
$$V_h = \left\{\mathbf{G} \in H(div, \Omega_1);\ \ \mathbf{G}_{/T} \in RT_0,\ \forall T \in \mathcal{T}_h^1\right\}.$$

Concerning the Lagrange multipliers on the interface, we introduce the space

$$K_h = \{\mu \in L^2(\Sigma)\,;\, \mu \in P_0(e),\ \forall e \in \mathcal{E}_h\}$$

and we put : $\mathbb{L}_h = (K_h)^2 \subset \mathbb{L}, \quad \mathbb{K}_h = (K_h)^3 \subset \mathbb{K}$.

We can now consider the following discrete version of (3.6) :

$$\left\{\begin{array}{llll} \text{Find } x_h \in \mathbb{X}_h^*, \Lambda_h \in \mathbb{L}_h & & & \\ \mathcal{A}_h(x_h, x') \quad + \quad \mathcal{I}(\Lambda_h, x') & = & \mathcal{F}_h(x'),\ \forall x' \in \mathbb{Y}_h \\ \mathcal{J}_h(\Lambda', x_h) & = & 0,\ \forall \Lambda' \in \mathbb{K}_h, \end{array}\right. \tag{4.1}$$

where the forms $\mathcal{A}_h(\cdot,\cdot)$ and $\mathcal{F}_h(\cdot)$ are obtained after an upwinding of the convective terms and where $\mathcal{J}_h(\cdot,\cdot)$ is deduced from $\mathcal{J}(\cdot,\cdot)$ by replacing ρ_2 by ρ_{2h}.

4.2. Well-posedness of the discrete problem. In order to establish the well-posedness of (4.1), we follow the mathematical analysis of the continuous coupled problem and we next establish the discrete versions of Lemmas 3.1-3.4, uniformly with respect to the discretisation parameter h. For this purpose, we need to prove first an auxiliary result, stated here below.

LEMMA 4.1. *Assume that $\exists\ \epsilon \in]0, \frac{1}{2}]$, such that any $\mathcal{T}_h^1$ satisfies the property :*

$$h^{\epsilon+\frac{1}{2}} \leq c\ h_{min,\Sigma}^{\epsilon}. \tag{4.2}$$

Then, for any $p \in M_h$ and $\theta \in K_h$, there exists $\mathbf{G} \in V_h$ satisfying :

$$\left\{\begin{array}{l} \mathbf{G}\cdot\mathbf{n} = \theta \ \ on\ \Sigma, \quad \mathbf{G}\cdot\mathbf{n} = 0 \ \ on\ \Upsilon_{\mathbf{G}} \setminus \Sigma \\ div\mathbf{G} = p \quad in\ \Omega_1. \end{array}\right. \tag{4.3}$$

Moreover, the next bound holds with c independent of h :

$$\|\mathbf{G}\|_{H(div,\Omega_1)} + \|\mathbf{G}\cdot\mathbf{n}\|_{0,\Sigma} \leq c(\|p\|_{0,\Omega_1} + \|\theta\|_{0,\Sigma}). \tag{4.4}$$

Proof. The idea is to define $\mathbf{G}$ as the Raviart-Thomas interpolate of a function $\mathbf{G}$ satisfying the above properties. Let us first note that θ belongs to $H^{\frac{1}{2}-\epsilon}(\Sigma)$ only, for any $0 < \epsilon \leq 1/2$. We regularize θ and we define $\tilde{\theta} \in H_0^1(\Sigma)$ such that :

$$\tilde{\theta} \in H_0^1(e) \text{ and } \int_e \tilde{\theta} d\sigma = \int_e \theta d\sigma, \quad \forall\ e \in \mathcal{E}_h.$$

More precisely, we can take $\tilde{\theta} = \theta\chi_e$ where χ_e is the bubble-function associated with the edge e satisfying $\chi_e \in P_2$ and $\int_e \chi_e d\sigma = h_e$. It is useful to note that :

$$\|\chi_e\|_{0,e} = c_0\ h_e^{1/2}, \quad |\chi_e|_{1,e} = c_1\ h_e^{-1/2}.$$

Then, we consider the following boundary value problem in the rectangle Ω_1 :

$$\begin{cases} \Delta\phi = p & \text{in } \Omega_1 \\ \frac{\partial\phi}{\partial n} = \tilde{\theta} & \text{on } \Sigma \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \Upsilon_{\mathbf{G}} \setminus \Sigma \\ \phi = 0 & \text{on } \Upsilon_p, \end{cases} \tag{4.5}$$

whose unique solution belongs to $H^2(\Omega_1)$ (Cf. [19]) and satisfies for any $0 < \epsilon \leq \frac{1}{2}$:

$$|\phi|_{\frac{3}{2}+\epsilon,\Omega_1} \leq c(\epsilon)\left(\|\Delta\phi\|_{-\frac{1}{2}+\epsilon,\Omega_1} + \|\partial_n\phi\|_{\epsilon,\Sigma}\right) \leq c(\epsilon)\left(\|p\|_{0,\Omega_1} + \left\|\tilde{\theta}\right\|_{\epsilon,\Sigma}\right).$$

Then we put $\mathbf{G} = E_h(\nabla\phi)$, where E_h is the Raviart-Thomas interpolation operator (cf. [17]). We recall that, for any $\mathbf{Q} \in H(div, T)$ with $\mathbf{Q}\cdot\mathbf{n} \in L^1(\partial T)$, $E_h(\mathbf{Q})$ is defined by the relations :

$$\int_e E_h(\mathbf{Q})\cdot\mathbf{n} d\sigma = \int_e \mathbf{Q}\cdot\mathbf{n} d\sigma, \ \ \forall e \subset \partial T.$$

Then one also has :

$$\int_T div E_h(\mathbf{Q}) dx = \int_T div\mathbf{Q} dx, \quad \forall\ T \in \mathcal{T}_h^1$$

so $\mathbf{G}$ obviously satisfies the relations (4.3). Since $div(\mathbf{G})$ and p, respectively $\mathbf{G}\cdot\mathbf{n}$ and θ are piecewise constant, one immediately gets :

$$\|div\mathbf{G}\|_{0,\Omega_1} = \|p\|_{0,\Omega_1},$$

$$\|\mathbf{G}\cdot\mathbf{n}\|_{0,\Sigma} = \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\left|\int_e \mathbf{G}\cdot\mathbf{n} d\sigma\right|^2\right)^{\frac{1}{2}} = \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\left|\int_e \tilde{\theta} d\sigma\right|^2\right)^{\frac{1}{2}} = \|\theta\|_{0,\Sigma}.$$

In view of establishing (4.4), one still has to bound $\|\mathbf{G}\|_{0,\Omega_1}$. It is classical that :

$$\forall\ T\ \in \mathcal{T}_h^1,\ \|\mathbf{G}\|_{0,T} \leq c\sum_{e\subset\partial T} h_e^{\frac{1}{2}}\|\mathbf{G}\cdot\mathbf{n}\|_{0,e} \leq c\sum_{e\subset\partial T} h_e^{\frac{1}{2}}\|\nabla\phi\|_{0,e}.$$

Thanks to the trace theorem and reverting to the reference element, one next obtains

$$\|\nabla\phi\|_{0,e} \leq c\left(h_T^{-\frac{1}{2}}\|\nabla\phi\|_{0,T} + h_T^{\epsilon}|\nabla\phi|_{\frac{1}{2}+\epsilon,T}\right).$$

Then by summing up on all $T \in \mathcal{T}_h^1$, it follows that :

$$\|\mathbf{G}\|_{0,\Omega_1} \leq c \left(|\phi|_{1,\Omega_1} + h^{\epsilon+\frac{1}{2}} \, |\phi|_{\frac{3}{2}+\epsilon,\Omega_1} \right), \quad \forall \; 0 < \epsilon \leq 1/2.$$

The weak formulation of (4.5) yields : $\quad |\phi|_{1,\Omega_1} \leq c \left(\|p\|_{0,\Omega_1} + \left\| \tilde{\theta} \right\|_{0,\Sigma} \right).$

Since $H^\epsilon(\Sigma)$ is the interpolate space of $L^2(\Sigma)$ and $H^1(\Sigma)$ (cf. [10]), we have :

$$\left\| \tilde{\theta} \right\|_{\epsilon,\Sigma} \leq c \left\| \tilde{\theta} \right\|_{0,\Sigma}^{1-\epsilon} \left\| \tilde{\theta} \right\|_{1,\Sigma}^{\epsilon} \leq c \left(\left\| \tilde{\theta} \right\|_{0,\Sigma} + \left\| \tilde{\theta} \right\|_{0,\Sigma}^{1-\epsilon} \left| \tilde{\theta} \right|_{1,\Sigma}^{\epsilon} \right).$$

Using that:

$$\left\| \tilde{\theta} \right\|_{0,\Sigma}^2 = \sum_{e \subset \Sigma} |\theta|^2 \, \|\chi\|_{0,e}^2 \leq c \sum_{e \subset \Sigma} h_e \, |\theta|^2 = c \, \|\theta\|_{0,\Sigma}^2 \,,$$

$$\left| \tilde{\theta} \right|_{1,\Sigma}^2 = \sum_{e \subset \Sigma} |\theta|^2 \, |\chi|_{1,e}^2 \leq c \sum_{e \subset \Sigma} \frac{1}{h_e} \, |\theta|^2 \leq \frac{c}{h_{min,\Sigma}^2} \, \|\theta\|_{0,\Sigma}^2 \,,$$

we finally obtain for any $0 < \epsilon \leq \frac{1}{2}$:

$$\|\mathbf{G}\|_{0,\Omega_1} \leq c(\epsilon) \left(\|p\|_{0,\Omega_1} + \|\theta\|_{0,\Sigma} + \frac{h^{\epsilon+\frac{1}{2}}}{h_{min,\Sigma}^{\epsilon}} \, \|\theta\|_{0,\Sigma} \right).$$

Therefore, estimate (4.4) holds if the condition (4.2) is checked. ∎

Remark 7. *In the limit case $\epsilon = 0$, the condition (4.2) is almost always satisfied. For $\epsilon = \frac{1}{2}$, it translates into $h_{min,\Sigma} \approx h^2$ which is not too restrictive.*

Lemma 4.2. *Assume (4.2). There exist $b_1^* > 0$, $b_2^* > 0$ independent of h such that :*

$$\forall \Lambda \in \mathbb{L}_h, \quad \sup_{x' \in \mathbb{Y}_h} \frac{\mathcal{I}(\Lambda, x')}{\|x'\|_{\mathbb{Y}}} \geq b_1^* \|\Lambda\|_{0,\Sigma},$$

$$\forall \Lambda' \in \mathbb{K}_h, \quad \sup_{x \in \mathbb{X}_h^0} \frac{\mathcal{J}_h(\Lambda', x)}{\|x\|_{\mathbb{X}}} \geq b_2^* \|\Lambda'\|_{0,\Sigma}.$$

Proof. Similarly to the proof of Lemma 3.1, with any $\Lambda = (\theta, \mu) \in \mathbb{L}_h$ we associate a vector function $x' \in \mathbb{Y}_h$ whose components are null except for $\mathbf{G}_1'$ and $\mathbf{q}_2'$. We take $\mathbf{q}_2' = \begin{pmatrix} \frac{r}{R^2}\mu \\ 0 \end{pmatrix}$ and $\mathbf{G}_1' = E_h(\nabla\phi)$ where ϕ is the solution of the auxiliary problem (4.5) with $p = 0$. Thanks to Lemma 4.1, we obtain :

$$\mathcal{I}(\Lambda, x') = \|\Lambda\|_{0,\Sigma}^2 \text{ and } \|x'\|_{\mathbb{Y}} \leq c\|\Lambda\|_{0,\Sigma}$$

with c independent of the discretisation, so the first inf-sup condition holds.

The proof of the second inequality is quite similar : with any $\Lambda' = (\theta', \zeta', \mu') \in \mathbb{K}_h$, we now associate $x \in \mathbb{X}_h^0$ whose components are null except for $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{q}_2$. One then chooses $\mathbf{q}_2 = \begin{pmatrix} \frac{r}{R^2}\mu' \\ 0 \end{pmatrix}$ and $\mathbf{G}_2 = \begin{pmatrix} \frac{r}{R^2}(\theta' - \zeta') \\ 0 \end{pmatrix}$ while $\mathbf{G}_1$ is taken as above, corresponding to θ'. This yields the announced result. ∎

Let us now introduce the discrete kernels of the bilinear forms $\mathcal{J}_h$ and $\mathcal{I}$:

$$\begin{aligned}\mathbb{J}_h^0 &= \left\{x \in \mathbb{X}_h^0\,;\ \mathcal{J}_h(\Lambda', x) = 0,\ \forall \Lambda' \in \mathbb{K}_h\right\},\\ \mathbb{I}_h &= \left\{x' \in \mathbb{Y}_h\,;\ \mathcal{I}(\Lambda, x') = 0,\ \forall \Lambda \in \mathbb{L}_h\right\},\end{aligned}$$

as well as the affine set : $\mathbb{J}_h^* = \{x \in \mathbb{X}_h^*\,;\ \mathcal{J}_h(\Lambda', x) = 0,\ \forall \Lambda' \in \mathbb{K}_h\}$.

Clearly, the elements of $\mathbb{J}_h^0$ satisfy :

$$\mathbf{G}_1 \cdot \mathbf{n}_{|e} = R\mathbf{G}_2 \cdot \mathbf{n}_{|e} = \frac{R\rho_{2h}}{|e|}\int_e \mathbf{u}_2 \cdot \mathbf{n} d\sigma, \quad \mathbf{q}_1 \cdot \mathbf{n}_{|e} = R\mathbf{q}_2 \cdot \mathbf{n}_{|e},\ \ \forall e \in \mathcal{E}_h \tag{4.6}$$

while those of $\mathbb{I}_h$ satisfy :

$$\mathbf{G}_1' \cdot \mathbf{n}_{|e} = \frac{R}{|e|}\int_e \mathbf{u}_2' \cdot \mathbf{n} d\sigma, \quad \mathbf{q}_1' \cdot \mathbf{n}_{|e} = R\mathbf{q}_2' \cdot \mathbf{n}_{|e},\ \ \forall e \in \mathcal{E}_h. \tag{4.7}$$

Thanks to Lemma 4.2, it is now sufficient to study the following discrete problem :

$$\left\{\begin{array}{l}\text{Find } x_h \in \mathbb{J}_h^*\\[2ex] \mathcal{A}_h(x_h, x') = \mathcal{F}_h(x'), \quad \forall x' \in \mathbb{I}_h.\end{array}\right. \tag{4.8}$$

It is then well-known that for any x_h solution of (4.8), there exists a unique multiplier $\Lambda_h \in \mathbb{L}_h$ such that the pair (x_h, Λ_h) satisfies the mixed problem (4.1).

As in the continuous case, problem (4.8) can be equivalently written as follows :

$$\left\{\begin{array}{ll}\text{Find } (\mathbf{U}_h, s_h) \in \mathbb{U}_h^* \times \mathbb{S}_h & \\ \mathtt{A}_h(\mathbf{U}_h, \mathbf{U}') + \mathtt{B}(s_h, \mathbf{U}') = \mathtt{F}_{1h}(\mathbf{U}'), & \forall \mathbf{U}' \in \mathbb{T}_h^0\\ \mathtt{B}(s', \mathbf{U}_h) - \mathtt{C}_h(s_h, s') = \mathtt{F}_{2h}(s'), & \forall s' \in \mathbb{S}_h\end{array}\right. \tag{4.9}$$

where $\mathtt{A}_h(\cdot,\cdot)$ now takes into account the convective term of the Navier-Stokes equations in the wellbore, while $\mathtt{C}_h(\cdot,\cdot)$ contains the convective terms coming from the energy equation in both the reservoir and the wellbore. We recall that all these convective terms are treated by upwinding schemes.

We can now establish the following preliminary results for problem (4.9).

Lemma 4.3. *Assume (4.2). There exist* $\beta_1^* > 0$, $\beta_2^* > 0$, *independent of* Δt *and* h, *such that :*

$$\forall s \in \mathbb{S}_h, \quad \sup_{\mathbf{U} \in \mathbb{U}_h^0} \frac{\mathtt{B}(s, \mathbf{U})}{\|\mathbf{U}\|} \geq \beta_1^* \|s\| \quad \textit{and} \quad \sup_{\mathbf{U}' \in \mathbb{T}_h^0} \frac{\mathtt{B}(s, \mathbf{U}')}{\|\mathbf{U}'\|} \geq \beta_2^* \|s\|.$$

Proof. The proof is similar to the one of Lemma 3.2. With any $s = (p_1, T_1, p_2, T_2) \in \mathbb{S}_h$, we associate a discrete function $\mathbf{U} = (\mathbf{G}_1, \mathbf{q}_1, \mathbf{G}_2, \mathbf{u}_2, \mathbf{q}_2) \in \mathbb{U}_h^0$ satisfying the discrete transmission conditions (4.6) on Σ, as well as:

$$\mathtt{B}(s, \mathbf{U}) \geq c\|s\|^2, \quad \|\mathbf{U}\| \leq c'\|s\|.$$

The component $\mathbf{u}_2$ is defined exactly as in the proof of the discrete inf-sup condition for the Navier-Stokes equations in the wellbore. Next, we put $\mathbf{G}_2 = \begin{pmatrix} \frac{r}{R}\overline{G}_{2r}(z) \\ 0 \end{pmatrix}$ where $\overline{G}_{2r}$ is null on $I \setminus \Sigma$ and piecewise constant on Σ, such that $\overline{G}_{2r} = \frac{\rho_{2h}}{|e|}\int_e \overline{u}_r d\sigma$ on every edge $e \in \mathcal{E}_h$.

In order to choose $\mathbf{G}_1$, we consider the solution ϕ of (4.5), with data $p = -p_1$ and $\theta_{/e} = \frac{R\rho_{2h}}{|e|}\int_e \overline{u}_r d\sigma$ and we put $\mathbf{G}_1 = E_h(\nabla\phi)$. Finally, we choose $\mathbf{q}_2$ associated with T_2 exactly as in the discrete wellbore problem and we put $\mathbf{q}_1 = E_h(\nabla\phi)$, where ϕ satisfies (4.5) with $p = T_1$ and $\theta = R\mathbf{q}_2 \cdot \mathbf{n}$. Thus, thanks to Lemma 4.1, the first inf-sup condition is checked. The proof of the second one is similar. ◻

Lemma 4.4. *There exists $\gamma^* > 0$, depending on Δt but independent of the discretisation, such that :*

$$\forall s \in \mathbb{S}_h, \ \ \mathtt{C}(s,s) \geq \gamma^*(\|p_1\|^2_{0,\Omega_1} + \|T_1\|^2_{0,\Omega_1} + \|T_2\|^2_{0,\Omega_2}).$$

A similar result holds for $\mathtt{C}_h(\cdot,\cdot)$, if Δt is now taken sufficiently small with respect to the discretisation parameter.

Proof. The first estimate for $\mathtt{C}(\cdot,\cdot)$ directly results from Lemma 3.3 with $\gamma^* = \gamma$, since $\mathbb{S}_h \subset \mathbb{S}$. The second one was already established when separately studying the discrete wellbore and reservoir models. Indeed, one can write that :

$$\mathtt{C}_h(s,s) = \mathtt{C}(s,s) + d_h(T_2,T_2) + D_h\left((p_1,T_1),(p_1,T_1)\right),$$

where d_h is positive and D_h satisfies :

$$D_h\left((p_1,T_1),(p_1,T_1)\right) \leq \frac{c}{h_1^2}\left\|\mathbf{G}_h^{n-1}\right\|_{0,\Omega_1}\left(\|p_1\|^2_{0,\Omega_1} + \|T_1\|^2_{0,\Omega_1}\right).$$

Then the result holds true since γ is proportional to $\frac{1}{\Delta t}$. ◻

Lemma 4.5. *Assume (4.2). For Δt sufficiently small, one has :*

$$\forall \mathbf{U} \in \mathbb{U}_h^0 \setminus \{0\}, \qquad \sup_{\substack{\mathbf{U}' \in \mathbb{T}_h^0 \\ \mathbf{U}-\mathbf{U}' \in Ker_h\mathtt{B}}} \frac{\mathtt{A}(\mathbf{U},\mathbf{U}')}{\|\mathbf{U}\|\|\mathbf{U}'\|} > 0.$$

A similar result holds for the bilinear form $\mathtt{A}_h(\cdot,\cdot)$, where Δt is now related to the discretisation parameter.

Proof. We closely follow the proof given at the continuous level in Lemma 3.4. We shall prove that there exist $c > 0$ and $\alpha^* > 0$ independent of the discretisation, such that for any $\mathbf{U} \in \mathbb{U}_h^0$, one can build $\mathbf{U}' \in \mathbb{T}_h^0$ satisfying :

$$\begin{aligned}
\mathtt{B}(s,\mathbf{U}) &= \mathtt{B}(s,\mathbf{U}'), \quad \forall s \in \mathbb{S}_h,\\
\|\mathbf{U}'\| &\leq c\,\|\mathbf{U}\|,\\
\mathtt{A}(\mathbf{U},\mathbf{U}') &\geq \alpha^*(\|\mathbf{G}_1\|^2_{0,\Omega_1} + \|\mathbf{q}_1\|^2_{0,\Omega_1} + \|\mathbf{G}_2\|^2_{H(div,\Omega_2)} + \|\mathbf{u}_2\|^2_{1,\Omega_2} + \|\mathbf{q}_2\|^2_{0,\Omega_2}).
\end{aligned}$$

For this purpose, let us take $\mathbf{U}' = (\mathbf{G}_1', \mathbf{q}_1, \chi, \mathbf{u}_2, \mathbf{q}_2)$ belonging to $\mathbb{T}_h^0$, where $\mathbf{G}_1'$ and χ are to be defined.

The norms $\|G_{2r}\|_{0,\Omega_2}$ and $\|u_r\|_{0,\Omega_2}$ are equivalent since one has :

$$\mathbf{G}_2 \cdot \mathbf{n}_{|e} = \frac{\rho_{2h}}{|e|}\int_e \mathbf{u}_2 \cdot \mathbf{n} d\sigma, \quad \forall e \in \mathcal{E}_h.$$

Then one can choose, as in the proof of Lemma 3.4, $\quad \chi = \epsilon(\partial_z G_{2z} - \frac{2}{R}\overline{G}_{2r})$.

In order to construct $\mathbf{G}_1'$, we consider the same boundary value problem as in Lemma 4.1, with data $p = 0$ and $\theta_{|e} = \frac{R}{|e|}\int_e (1-\rho_{2h})\mathbf{u}_2 \cdot \mathbf{n} d\sigma, \quad \forall e \in \mathcal{E}_h$ and we put $\mathbf{G}_1' = E_h(\nabla\psi) + \mathbf{G}_1$. The above choice ensures that $div\mathbf{G}_1' = div\mathbf{G}_1$ as well as

$$(\mathbf{G}_1' \cdot \mathbf{n})_{|e} = (\mathbf{G}_1 \cdot \mathbf{n})_{|e} + \frac{1}{|e|}\int_e \frac{\partial\psi}{\partial n} d\sigma = \frac{R}{|e|}\int_e \overline{u}_r d\sigma, \ \ \forall e \in \mathcal{E}_h$$

and moreover :

$$\|\mathbf{G}_1'\|_{0,\Omega_1} \leq c(\|\mathbf{G}_1\|_{0,\Omega_1} + \|u_r\|_{0,\Omega_2})$$

which allows us to conclude. □

We are now able to establish the well-posedness of the discrete problem (4.8), and implicitly of (4.1). Let us first recall that both the discrete reservoir and wellbore models have unique solutions if Δt satisfies :

$$\Delta t \leq \min(C_1 h^2_{min,\Omega_1}, C_2 h_{min,\Omega_2}) \tag{4.10}$$

with $h_{min,\Omega_1} = \min_{T\in\mathcal{T}_h^1} h_T,\ h_{min,\Omega_2} = \min_{T\in\mathcal{T}_h^2} h_T$ and with C_1, C_2 independent of the discretisation. Then we get :

THEOREM 4.6. *Assume (4.2). Then problem (4.8) has a unique solution, for Δt satisfying (4.10).*

Proof. Due to the finite dimensional framework, it is sufficient to prove the uniqueness of the solution. The proof is obvious thanks to lemmas 4.3, 4.4 and 4.5 : the positivity of $\mathtt{A}_h(\cdot,\cdot)$ and $\mathtt{C}_h(\cdot,\cdot)$ gives that the solution of the homogeneous discrete problem satisfies $\mathbf{U}_h = \mathbf{0}$, $p_{1h} = T_{1h} = T_{2h} = 0$ while the discrete inf-sup condition on $\mathtt{B}(\cdot,\cdot)$ implies $p_{2h} = 0$. □

4.3. Existence of a solution for the continuous problem. Finally, let us now prove the existence of a solution in the continuous case.

THEOREM 4.7. *Assume (4.2) and (4.10). The continuous coupled problem (3.6) with $\alpha = 0$ (that is, without convection in the energy laws) has at least one solution.*

Proof. As already mentioned, we apply a Galerkin method. We first consider a sequence of approximated problems of (3.6), written on the finite dimensional spaces previously introduced :

$$\left\{\begin{array}{llll} \text{Find } \widetilde{x}_h \in \mathbb{X}_h^*, \widetilde{\Lambda}_h \in \mathbb{L}_h & & & \\ \mathcal{A}(\widetilde{x}_h, x') & + \quad \mathcal{I}(\widetilde{\Lambda}_h, x') & = & \mathcal{F}(x'),\ \forall x' \in \mathbb{Y}_h \\ \mathcal{J}_h(\Lambda', \widetilde{x}_h) & & = & 0,\ \forall \Lambda' \in \mathbb{K}_h \end{array}\right. \tag{4.11}$$

where in the definition of $\mathcal{J}_h(\cdot,\cdot)$, $\widetilde{\rho_{2h}}$ now stands for the piecewise constant $L^2(\Sigma)$-orthogonal projection of ρ_2. The four previous Lemmas imply that each discrete problem (4.11) has a unique solution $(\widetilde{x}_h, \widetilde{\Lambda}_h)$.

According to Lemmas 4.4 and 4.5, the discrete solution $\widetilde{x}_h = (\widetilde{\mathbf{U}}_h, \widetilde{s}_h)$ satisfies the following estimate, uniformly with respect to h :

$$\begin{aligned} \left\|\widetilde{\mathbf{G}}_{1h}\right\|_{0,\Omega_1} + \|\widetilde{\mathbf{q}}_{1h}\|_{0,\Omega_1} + \left\|\widetilde{\mathbf{G}}_{2h}\right\|_{H(div,\Omega_2)} + \|\widetilde{\mathbf{u}}_{2h}\|_{1,\Omega_2} + \|\widetilde{\mathbf{q}}_{2h}\|_{0,\Omega_2} \\ + \|\widetilde{p}_{1h}\|_{0,\Omega_1} + \left\|\widetilde{T}_{1h}\right\|_{0,\Omega_1} + \left\|\widetilde{T}_{2h}\right\|_{0,\Omega_2} \leq c. \end{aligned}$$

The inf-sup condition on $\mathtt{B}(\cdot,\cdot)$ ensures, cf. Lemma 4.3, that $\widetilde{p}_{2h}$ is bounded in $L^2(\Omega_2)$, since :

$$\begin{aligned} \beta_2^* \|\widetilde{p}_{2h}\|_{0,\Omega_2} &\leq \sup_{\mathbf{U}'\in\mathbb{T}_h^0} \frac{\mathtt{B}(\widetilde{s}_h, \mathbf{U}')}{\|\mathbf{U}'\|} = \sup_{\mathbf{U}'\in\mathbb{T}_h^0} \frac{\mathtt{A}(\widetilde{\mathbf{U}}_h, \mathbf{U}') - \mathtt{F}_1(\mathbf{U}')}{\|\mathbf{U}'\|} \\ &\leq c\left(\left\|\widetilde{\mathbf{G}}_{1h}\right\|_{0,\Omega_1} + \|\widetilde{\mathbf{q}}_{1h}\|_{0,\Omega_1} + \left\|\widetilde{\mathbf{G}}_{2h}\right\|_{H(div,\Omega_2)} + \|\widetilde{\mathbf{u}}_{2h}\|_{1,\Omega_2} + \|\widetilde{\mathbf{q}}_{2h}\|_{0,\Omega_2}\right). \end{aligned}$$

Next, by choosing $p_1' = div\widetilde{\mathbf{G}}_{1h}$, $T_1' = div\widetilde{\mathbf{q}}_{1h}$, $p_2' = 0$ and $T_2' = \partial_z \widetilde{q}_{zh} + \frac{2}{R}\widetilde{\widetilde{q}}_{rh}$ as test-function s' in the second variational equation, one gets :

$$\|div\widetilde{\mathbf{G}}_{1h}\|^2_{0,\Omega_1} + \|div\widetilde{\mathbf{q}}_{1h}\|^2_{0,\Omega_1} + \|\frac{1}{r}div(r\widetilde{\mathbf{q}}_{2h})\|^2_{0,\Omega_2} = \mathtt{F}_2(s') - \mathtt{C}(s'\widetilde{s}_h) \leq c\,\|s'\|\,.$$

Using next that

$$div\widetilde{\mathbf{q}}_{2h} = \frac{1}{r}div(r\widetilde{\mathbf{q}}_{2h}) - \frac{1}{R}\widetilde{\widetilde{q}}_{rh},$$

one can now conclude that $div\widetilde{\mathbf{G}}_{1h}$, $div\widetilde{\mathbf{q}}_{1h}$, and $div\widetilde{\mathbf{q}}_{2h}$ are also uniformly bounded with respect to the L^2-norm.

The second equation of problem (4.11) gives that $\widetilde{\mathbf{G}}_{1h}\cdot\mathbf{n}$ and $\widetilde{\mathbf{q}}_{1h}\cdot\mathbf{n}$ are uniformly bounded in $L^2(\Sigma)$, since :

$$\widetilde{\mathbf{G}}_{1h}\cdot\mathbf{n}_{|e} = \frac{R\rho_{2h}}{|e|}\int_e \widetilde{u}_{rh}d\sigma, \qquad \widetilde{\mathbf{q}}_{1h}\cdot\mathbf{n}_{|e} = R(\widetilde{\widetilde{q}}_{rh})_{|e}, \qquad \forall e \in \mathcal{E}_h$$

and $\widetilde{u}_{rh}$, $\widetilde{\widetilde{q}}_{rh}$ are both bounded in $L^2(\Sigma)$.

So the sequence $(\widetilde{x}_h)_h$ is bounded in the $\mathbb{X}$-norm, whereas the uniform inf-sup condition satisfied by $\mathcal{I}(\cdot,\cdot)$ (cf. Lemma 4.2) implies that $(\widetilde{\Lambda}_h)_h$ is bounded in the $\mathbb{L}$-norm. Therefore, one can extract a subsequence, still denoted by $(\widetilde{x}_h, \widetilde{\Lambda}_h)_h$, weakly convergent in the space $\mathbb{X}\times\mathbb{L}$ towards $(\widetilde{x}, \widetilde{\Lambda})$. Due to the approximation properties of the finite element spaces employed, one has that for any $(x', \Lambda') \in \mathbb{Y}\times\mathbb{K}$, there exists a sequence $(x'_h, \Lambda'_h) \in \mathbb{Y}_h \times \mathbb{K}_h$ strongly convergent towards (x', Λ'). Moreover, $\rho_{2h}\theta'_h$ strongly converges towards $\rho_2\theta'$ in $L^2(\Sigma)$, too.

Finally, a classical passage to the limit in (4.11) yields that the weak limit $(\widetilde{x}, \widetilde{\Lambda})$ is in fact a solution of problem (3.6), which ends the theorem proof. □

REMARK 8. *One may equally prove that the continuous problem with convection (i.e. $\alpha = 1$) also has a unique solution for a sufficiently small time-step, by using the regularity of the solution of (3.6) together with Fredholm's alternative (see the analysis of the separate reservoir and wellbore models, cf. [1] and [2]).*

5. Numerical results. We present in this section some numerical tests in order to validate our coupled code from both numerical and physical points of view. Firstly, we are interested in the convergence of the solution with respect to mesh refinement. Secondly, we treat a real case in order to compare the results given by the coupled code with those obtained by the sole reservoir and wellbore simulators.

5.1. Mesh convergence. We consider here a two-layer reservoir where only the lower layer is perforated. The reservoir is associated with a wellbore and is characterized by homogeneous properties. We have deliberately reduced the dimensions of the reservoir (length=10m, width=2m), in order to avoid considerable calculations.
The production of a light oil is simulated during 7 days by imposing a constant flowrate (of 1500 m^3/day) at the pipe's surface and a constant pressure on the external boundary. The fluid's viscosity in the pipe is about $8\times 10^{-4} Pa.s$.
In what follows, our aim is to study the behavior of the pressure and of the temperature with respect to mesh refinement. For this purpose, we consider congruent meshes $\mathcal{T}_{\frac{h}{i}}$, $i \in \{2, 4, 8\}$ obtained from an initial mesh $\mathcal{T}_h$ as follows: every triangle in the reservoir is divided into four congruent ones and every rectangle in the well into 2 congruent ones. Note that the refinement in the wellbore takes place only with

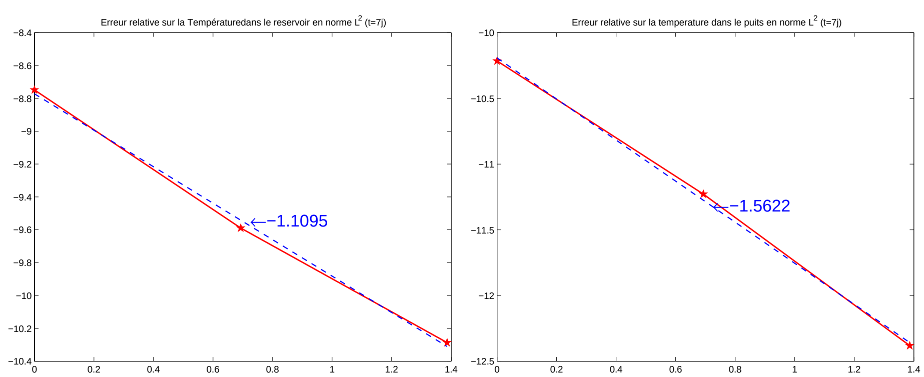
Mesh	Nodes	Edges	Triangles
Mesh $\mathcal{T}_h$	86	217	132
Mesh $\mathcal{T}_{h/2}$	303	830	528
Mesh $\mathcal{T}_{h/4}$	1133	3244	2112
Mesh $\mathcal{T}_{h/8}$	4377	12824	8448

TABLE 5.1
Congruent meshes for the reservoir

respect to the vertical direction.
For each intermediate mesh, we evaluate the L^2-norm of the error between the current solution and the one obtained on the finest mesh $\mathcal{T}_{\frac{h}{8}}$ (chosen as a reference solution). We have represented in Figure 5.1 and Figure 5.2 the logarithm of the error in terms of $log(1/h)$, for the pressure and the temperature. We numerically obtain :

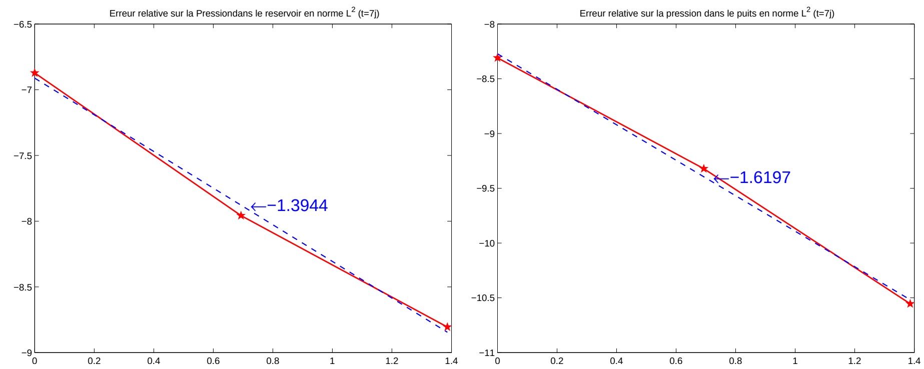
$$\|T - T_h\|_{0,\Omega} \leq Ch^{\alpha},$$

with α approximately equal to 1.1 in the reservoir and to 1.5 in the well. Similar results hold for the pressure, cf. Figure 5.2.



(a) order of the L^2-error in the reservoir (b) order of the L^2-error in the well

FIG. 5.1. *Convergence rate for the temperature at t=7 days.*



(a) order of the L^2-error in the reservoir (b) order of the L^2-error in the well

FIG. 5.2. *Convergence rate for the pressure at t=7 days.*

5.2. A more realistic application. The separate reservoir and wellbore simulators were previously validated from a numerical and a physical point of view (see [1], [2]), including comparisons with recorded pressure and temperature data and with a well-test software PIE (cf. *www.welltestsolution.com*). Therefore, our goal is to compare the results obtained by the coupled code with those given by the separate codes, in order to validate our simulator.

We treat here the case of a realistic reservoir divided into seven geological layers, where only three of them are perforated. The reservoir is characterized by highly heterogeneous physical properties (cf. Figure 5.3) and is fed by imposing a constant pressure $p_\gamma = 400\ bar$ on the external boundary. The reservoir is 50m large and 20m high. The respective heights of the layers are, from the top to the bottom : 5.5m, 3.2m, 1.5m, 2.7m, 1.7m, 2.3m and 3.1m, whereas the associated wellbore is only 0.15m large but 70m high.

We simulate the production of a light oil during 28 days for the coupled problem, as well as for the sole reservoir and wellbore problems. When dealing with the coupled code, we impose a constant flowrate $Q = 6500\, m^3/day$ at the pipe's surface while when treating the reservoir, a difference of pressure $\Delta p = 10\ bar$ between the perforations and the external boundary is set. All data are realistic. An adiabatic condition $\mathbf{q} \cdot \mathbf{n} = 0$ is also imposed on the external boundary of the reservoir. When computing the wellbore model, we impose as boundary conditions on the perforations the values given by the reservoir code.

Concerning the time stepping, we have noticed that the condition of sufficiently small Δt, required by certain theoretical results, does not seem to influence the performance of the code. A variable time step can be chosen during a simulation. In practice, rather small time steps (of about one hour) are taken during the transitory regime, whereas larger ones (of about one day) can be imposed once the flow has reached the steady state.

Let us now compare the results of the coupled code with those of the reservoir code. One can first see in Figure 5.4 that the flowrate imposed at the top of the well in the coupled model yields a difference of pressure $\Delta p \simeq 10\ bar$, which coincides with that imposed as boundary condition in the sole reservoir model. Concerning the temperature, an increase due to the Joule-Thomson effect is noticed in the two cases. The graphics obtained by the two simulators are very similar, as one can see in Figure 5.6.

As regards the comparison with the sole wellbore code, Figure 5.5 shows very similar results for G_z (from which one computes the production flowrate in the well by means of a compatibility condition). Thus, we obtain a flowrate Q for the sole wellbore model close to that imposed as boundary condition in the coupled problem.

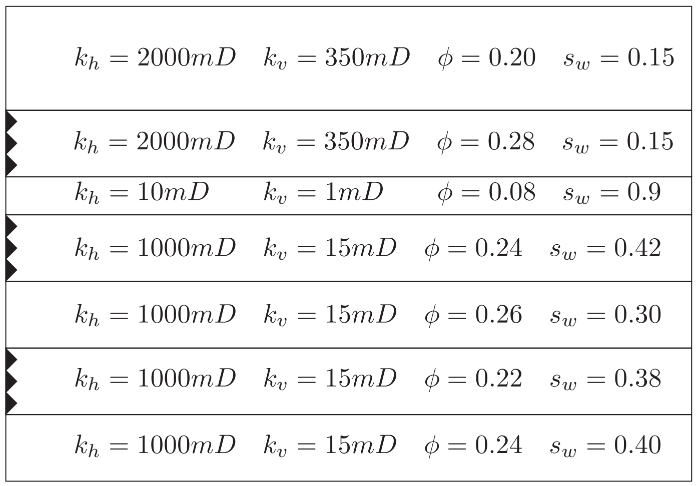
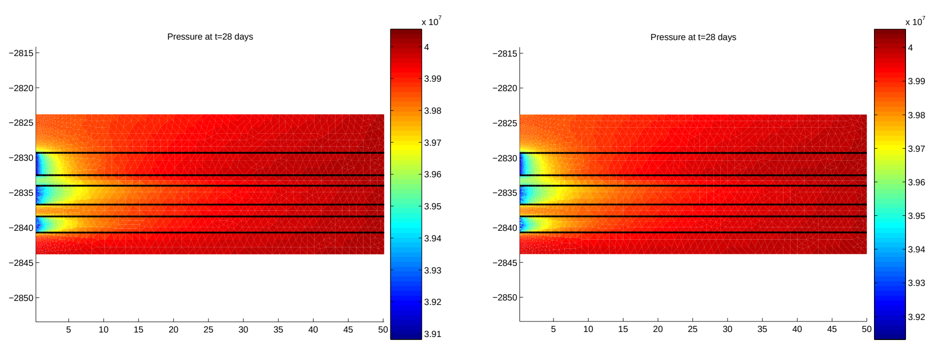
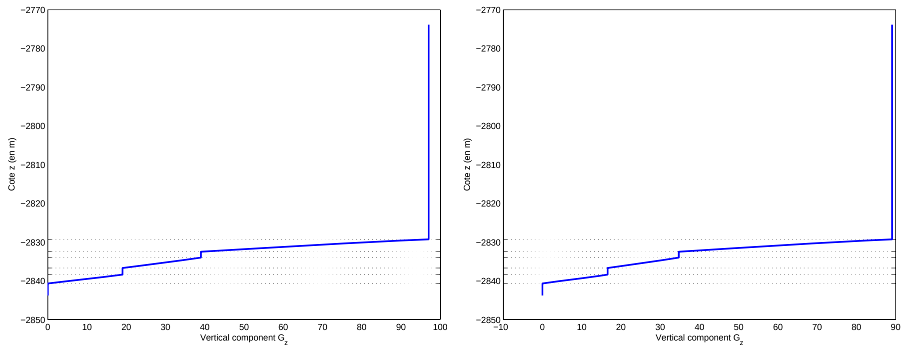


FIG. 5.3. *Longitudinal section of the reservoir.*



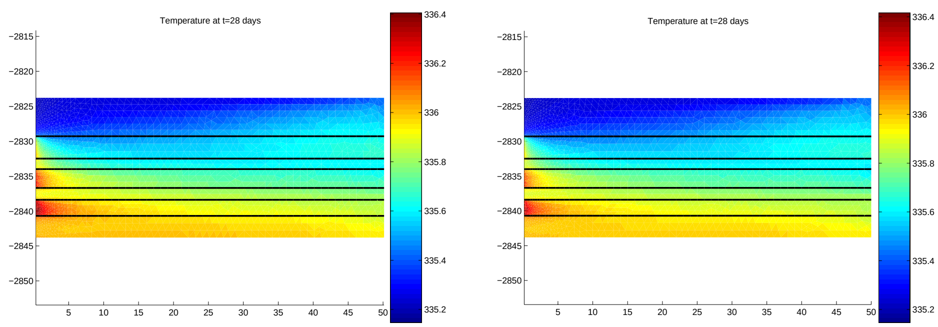
(a) Pressure given by reservoir code (b) Pressure given by coupled code

FIG. 5.4. *Comparison of the pressure maps in the reservoir at t = 28 days.*



(a) G_z given by wellbore code (b) G_z given by coupled code

Fig. 5.5. *Comparison of the vertical mass fluxes in the wellbore at t = 28 days.*

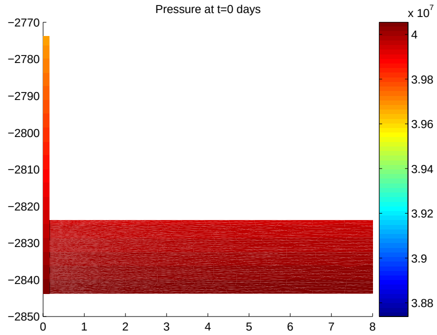


(a) Temperature given by reservoir code (b) Temperature given by coupled code

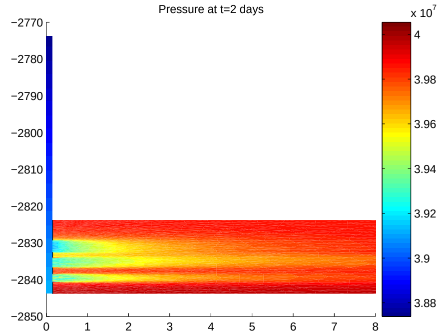
Fig. 5.6. *Comparison of the temperature maps after one-month production.*

We next observe the evolution of both the reservoir and the well during a one-month production. One can see in Figures 5.7, 5.8 and 5.9 the maps for the pressure, the temperature, respectively the density computed by the coupled code. Besides the initial and the final time-steps, we have chosen to represent the maps at t=2 and t=7 days since afterwards the flow almost reaches the steady state. The above mentioned figures focus on the neighbourhood of the perforations since due to the large aspect ratio between the reservoir and the wellbore, we only visualise 8m of the reservoir in the radial direction.

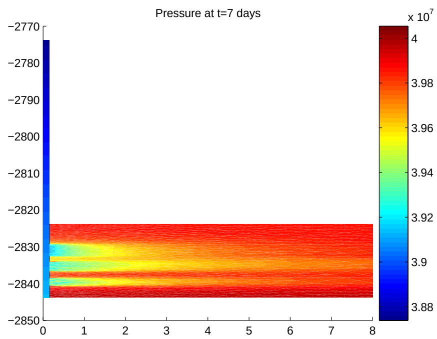
The numerical results for the previous quantities correspond to the physical behavior expected by petroleum engineers. Moreover, one may note that the transmission conditions at the interface are satisfied: the temperature takes the same values in the wellbore and in the reservoir while the pressure is slightly different in the wellbore (according to the relation $p_2 - \tau_{rr} = p_1$).



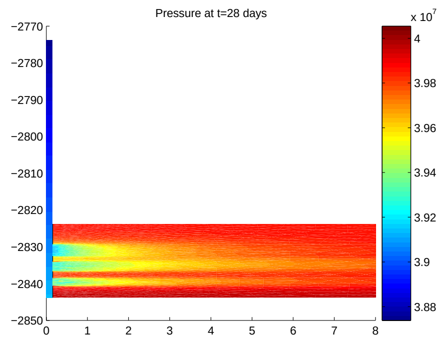
(a) Pressure at t=0 day



(b) Pressure at t=2 days

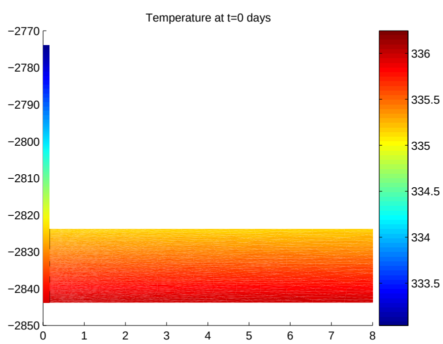


(c) Pressure at t=7 days

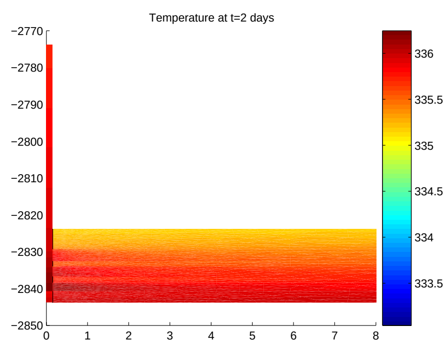


(d) Pressure at t=28 days

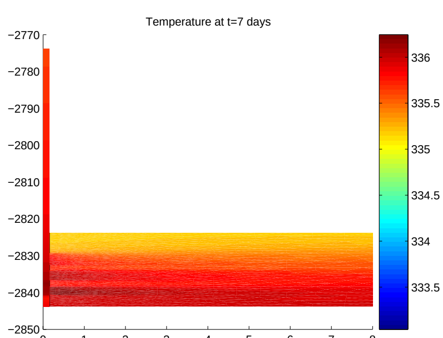
FIG. 5.7. *Behavior of the pressure during a one month production.*



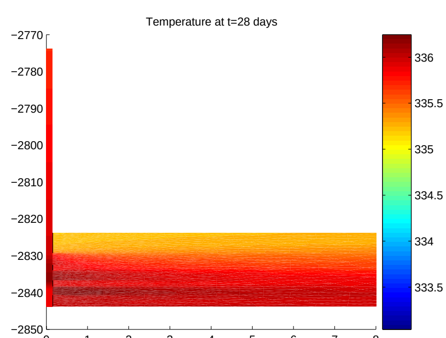
(a) Temperature at t=0 day



(b) Temperature at t=2 days



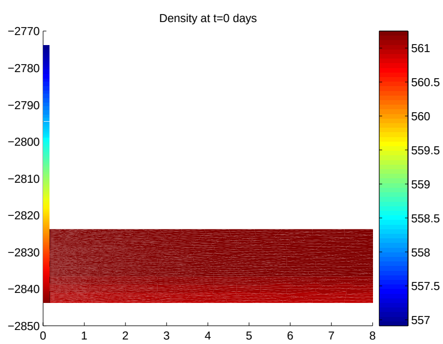
(c) Temperature at t=7 days



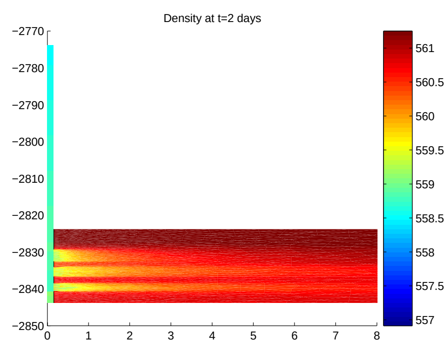
(d) Temperature at t=28 days

FIG. 5.8. *Behavior of the temperature during a one month production.*

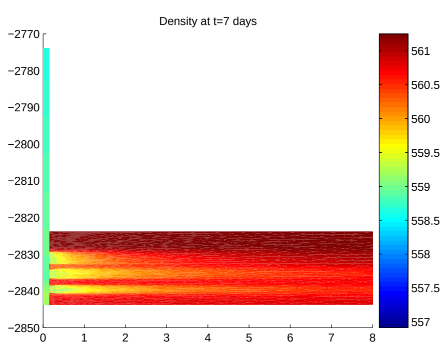
Finally, we also show the specific flux **G** for the previous test-case. As one can see in Figure 5.10(a), the computed velocity in the wellbore is much more important than the velocity in the reservoir. This is due to the fact that for a given cell in the wellbore, the flux is obtained by summing up the contributions of all the lower perforations. In order to better visualise the flow near the perforations, we next apply different scalings in the two domains (of ratio equal to 10). The corrresponding fluxes can be seen in Figure 5.10(b).



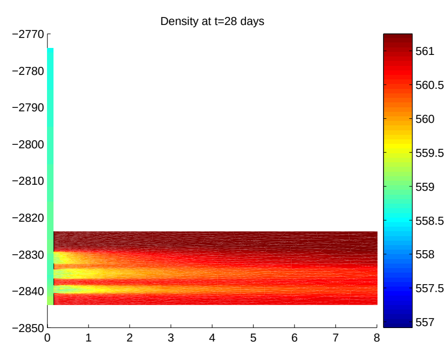
(a) Density at t=0 day



(b) Density at t=2 days

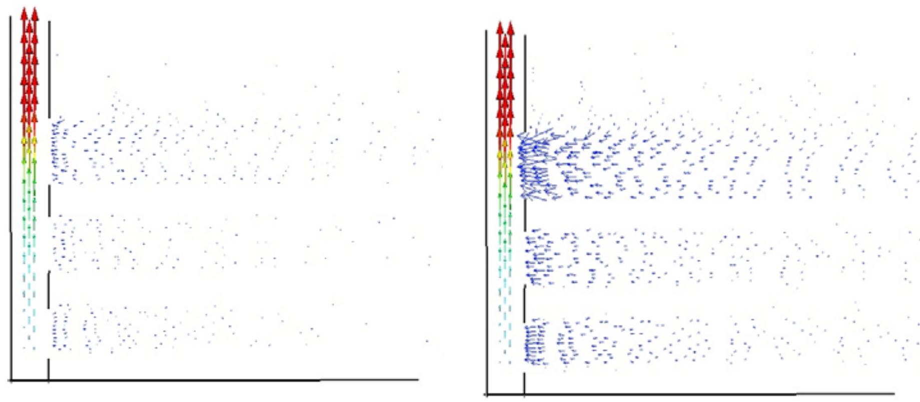


(c) Density at t=7 days



(d) Density at t=28 days

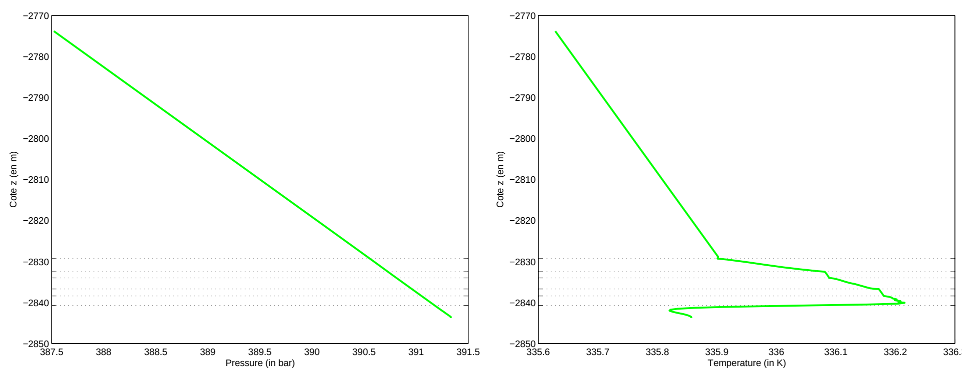
FIG. 5.9. *Behaviour of the density during a one month production.*



(a) Same scale in the 2 domains (b) Different scales in each domain

Fig. 5.10. *Specific flux at the end of the production.*

As regards the wellbore results, it is important to notice that we recover the well-known fact that the pressure is primarily influenced by the gravity. It goes the same way for the temperature above perforations, as one may see in Figure 5.11.



(a) Pressure (b) Temperature

Fig. 5.11. *Pressure and temperature in the wellbore at the end of the production.*

Acknowledgments. The authors thank the two referees for the constructive remarks.

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